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THE TOPOLOGY OF THE CALCULUS OF VARIATIONS IN THE LARGE

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THE TOPOLOGY OF THE CALCULUS OF VARIATIONS IN THE LARGE

by

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Foreword

Quite some time ago the connection was observed between the topological properties of a manifold and the character of the critical points of an arbitrary function f on it; that is, the points at which $df = 0$. We must add that the words "number of critical points" may be understood in two ways. One way is the sum of the multiplicities of the critical points. This is the so-called number of analytically distinct critical points. Another way is the number of geometrically distinct critical points. It helps in understanding this situation to note that the number of analytically distinct roots of an algebraic equation of the nth degree, i.e., the sum of the multiplicities of the roots, is equal to n, while the number of geometrically distinct roots may be no larger than unity. Correspondingly, various topological invariants have been applied to estimate the number of geometrically and analytically distinct roots.

The shift of attention away from the critical points of a function defined on a manifold towards the corresponding variational problems, i.e., towards the extremals of functionals defined on the function-space of admissible curves of a given variational problem, leads to a shift from the topological investigation of n-dimensional manifolds to that of more general abstract spaces. Thus the problems of analysis become intertwined with the investigations of modern-day abstract topology.

In our survey paper, published in "Uspehi Matematičeskih Nauk" (Ljusternik [**16**]), we discussed the leading applications of topology to research in the calculus of variations. That paper should be considered as an introduction to the present paper. Here we restrict ourselves to an indication of the most important results.

Marston Morse and his students investigated the connection between Betti numbers modulo 2 and the character and number of analytically distinct critical points. The basic results of Morse are presented in his significant book "The Calculus of Variations in the Large". Morse also applies his methods to functional spaces, making concrete studies of the Betti numbers modulo 2 of the space of rectifiable curves with common endpoints on the n-dimensional sphere and of the space of closed rectifiable curves on it. The most important result of Morse was the local existence of a countable sequence of geodesic arcs joining two given points on a manifold homeomorphic to the n-dimensional sphere.

The Moscow mathematicians Ljusternik, Šnirel′man and their students investigated the connection between the number of geometrically distinct critical points and the topological properties of manifolds. In this connection Ljusternik introduced the homotopy invariant "category" [**16**], estimating the number of geometrically distinct critical points. In turn, in order to estimate this invariant, which is difficult to calculate, Šnirel′man [**24**] introduced homology invariants which turned out to be closely connected with the ring of intersections of a manifold. Later, these invariants were studied, along with new homotopy invariants, by Èl′sgol′c [**29**]. Frolov and Èl′sgol′c [**30**] also introduced a new and computationally very convenient "length" of a manifold.

A very important tool in these investigations was the theorem of L. S. Pontrjagin on the removal of a cycle [**23**].

Homotopy invariants such as category may also be defined directly for functional spaces. In particular cases it was possible to compute such invariants and thus to investigate the number of solutions of the corresponding variational problem. It was by such a method that the existence of three closed geodesics on a surface of genus zero was proved by Ljusternik and Šnirel′man in [**19**]. However, in other cases it was not possible to give direct definitions of homotopy invariants.

In the present paper we investigate the homological properties of functional spaces in connection with the estimation of the number of solutions of the corresponding variational problems. These properties are closely related, on the one hand, to previous work of Moscow mathematicians on topological methods in variational problems, and on the other hand to the investigations of the Moscow topological school on the homology theory of abstract spaces. In a certain sense this paper constitutes a synthesis of both directions of the Moscow topological school.

In the most important papers on the calculus of variations these functional spaces are usually compact or close to compact. The basic concepts and facts of combinatorial topology are carried over to such spaces. Here, we must mention first of all the work of P. S. Aleksandrov. In particular, in [2a] he transferred the basic theorems of duality to such spaces, which then found their application in the calculus of variations in the large.

The upper cycles in functional spaces and their products, introduced into topology by A. N. Kolmogorov and P. S. Aleksandrov, were an essential tool in this research. Indeed, in the case of infinite-dimensional spaces the upper images yield geometrical objects that are new in principle (they are infinite-dimensional with a finite-dimensional defect relative to the whole space) which are just as necessary for studying the properties of the whole space as are the usual finite-dimensional "lower images". In concrete investigations the upper cycles in particular functional spaces admit intuitive geometrical definition in the form, for example, of the family of curves passing through given points of the manifold, or of the space of curves on that manifold and so forth.

Chapter I of this paper presents general methods in the calculus of variations in the large. It formulates general principles making it possible to determine when infinite families of extremals appear and to estimate the number of solutions of a variational problem in connection with the ring of products in the corresponding functional space.

Chapter II gives applications of the general methods to the theory of eigenvalues of certain classes of nonlinear integral equations.

Chapter III investigates "critical sets", consisting of geodesic curves on manifolds. In addition, we reproduce the author's theorem that each compact family of non-selfintersecting curves on the two-dimensional sphere can be reduced to a family of circumferences by a continuous deformation.

Chapter IV investigates the functional space of all rectifiable curves on the sphere with common endpoints (admissible curves of a variational problem with fixed endpoints). Here we first give a complete investigation of the topological structure of an essentially nonlinear functional space. We construct upper and lower homology groups and the ring of intersections (for various coefficient groups). We also give investigations of certain homotopy properties of this space. As an application we present a complete investigation of the geodesic arcs joining two given points on a surface of genus 0, sharpening some of Morse's results.

In Chapter V general methods are used to study closed geodesics on manifolds homeomorphic to the n-dimensional sphere. For the case $n = 2$ we give a new proof of a theorem earlier proved by the author simultaneously with L. G. Šnirel'man on the existence of three closed geodesics. For the case $n > 2$ under certain metric restrictions this theorem can be generalized: one proves the existence of at least $(n + 1)$ closed geodesics.

For the case $n = 2^k$ we prove the presence of $(2n + 1)$ such geodesics.

In conclusion, we remark that the study of the homological properties of concrete infinite-dimensional functional spaces is a highly interesting problem in topology.

CHAPTER I

General Principles of the Calculus of Variations in the Large

1. Critical values.

K-sets. In the investigation of variational problems it is necessary to separate out special classes of sets; for example, sets on which the functional is continuous, sets of admissible elements under special restrictions in certain problems, and so forth.

Suppose given a metric space M. We shall consider a class $[A]$ of closed sets on M, with the following property: if the set A belongs to the class $[A]$, then any closed subset belongs to the same class. The class $[A]$ will be called a K-class, and the sets of K-classes $[A]$ will be called K-sets.

For example, suppose that J is a functional given in M. We may define the K-class $[A]$ of those closed sets A on which J is continuous. We may also define the K-class $[B]$ of those closed sets B which have finite dimension.

Continuous deformations may be defined in the usual way. Suppose that A is some set (in general, defined abstractly) with a topology defined on it. To each point a of the set A and to each value of the parameter t of the segment $[0, 1]$ there corresponds a point $f(a, t)$ of the space M, while $f(a, t)$ is continuous relative to a and relative to t.

The family of points of the form $f(a, t)$ with a fixed t forms a set A_t, namely a continuous image of the set A. The operation thus defined, which transforms the set A_0 into the set A_1 is called a *deformation* of A_0, as an image of the set A.

If the set A_0 coincides with A and $f(a, 0) = a$, we call such an operation a *simple deformation* of A.

Along with classes of K-sets we consider certain classes of deformations which we call *K-deformations.* Here the successive application of two K-deformations, one after the other, is again a K-deformation, and a K-deformation maps a K-set B_0 again into a K-set B_1. However, the intervening stages of the deformation of B_0 may fail to be K-sets. The identical deformation is always considered to be a K-deformation.

For example, on a straight line we may define a K-class $[A]$, namely as the class of sets A not containing some point a, and a class of K-

deformations of these sets, i.e., namely the class of shifts along the line (as a "rigid body") as a result of which the K-set A (not containing the point a) goes into a K-set (however, at certain moments during the shift the image of the set A may indeed go through a). For example, in the space of all closed curves on the sphere we shall later consider K-sets consisting of curves without double points and K-deformations of such sets, transforming them anew into sets of curves without double points, i.e., into K-sets (though at intervening moments of the deformation the K-sets may fail to be K-sets).

K-critical points. We shall say that a set A compact in itself, empty or nonempty, on a level hypersurface $(I = c)$ (the set of points at which the functional $I = c$) *possesses the K-property* if for any sufficiently small $\alpha > 0$ there exists an $\epsilon = \epsilon(\alpha) > 0$, where $\epsilon(\alpha) \to 0$ as $\alpha \to 0$, such that any K-set of $(I \leqq c + \alpha)$ may be carried by a K-deformation into the region $(I \leqq c - \alpha) + S(A, \epsilon)$. We shall say in this case that $(I = c)$ itself possesses this property.

EXAMPLE 1. Suppose that M is a compactum and I a continuous function on it. The level hypersurface $(I = c)$, which we shall denote by A_c, has the following easily proved property: for any $\alpha > 0$ there exists an $\epsilon = \epsilon(\alpha)$ with $\epsilon(\alpha) \to 0$ as $\alpha \to 0$ such that if for the point x we have $| I(x) - c | < \alpha$, then $x \in S(A_0, \epsilon)$. It follows that $(I \leqq c + \alpha) \subset (I \leqq c - \alpha) + S(A_c, \epsilon)$. Since the identity operation is a K-operation and A_c is a set compact in itself, it therefore follows that A_c has the K-property.

If for the functional I the entire level surface $(I = c)$ has the K-property, then *the functional I itself has the K-property.* For example, all continuous functionals on compacta possess the K-property. In §§11 and 12 we show that the functional defining the length of a curve in the space R of rectifiable arcs on the sphere with common endpoints and in the space P of rectifiable closed curves on the sphere has the K-property.

Suppose that the functional I, and therefore the entire hypersurface $(I = c)$, has the K-property. Two cases arise:

(1) The empty set has the K-property on $(I = c)$. This means that for any sufficiently small $\alpha > 0$ each K-set of $(I \leqq c + \alpha)$ may be carried by a K-deformation into the region $(I \leqq c - \alpha)$. In this case c will be called a K-ordinary value.

(2) Only certain nonempty, compact in themselves, sets A have the K-property on $(I = c)$. In this case there exists on $(I = c)$ a nonempty set A_0 compact in itself which is irreducible with respect to the property of possessing the K-property. Such a set will be called K-critical.

A set A_0 compact in itself is called K-critical on $(I = c)$ if it is a minimal set with respect to the following property: for each $\epsilon > 0$ there exists an $\epsilon = \epsilon(\alpha) > 0$ with $\epsilon(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$, such that any K-set in $(I \leqq c + \alpha)$ may be carried into $(I \leqq c + \alpha) + S(A_0, \epsilon)$. In this case c is called *a K-critical value.*

It c is not a K-critical value, then we shall call it *a K-ordinary value.*

The points of a K-critical set will be called *K-critical* points.

If the K-sets are arbitrary closed sets, and the K-deformations are arbitrary deformations of them, then we shall speak simply of critical sets, points, and values, instead of K-critical, and correspondingly for ordinary values.

EXAMPLE 2. Consider the well-studied case that f is a twice-differentiable function on a twice-differentiable Riemann n-dimensional manifold R_n.

If $(f = c)$ contains no stationary points, i.e., points at which $df = 0$, then any closed set of $(f \leqq c + \alpha)$ may for sufficiently small $\alpha > 0$ be carried into $(f \leqq c - \alpha)$, i.e., c is an ordinary value.

If $(f = c)$ contains a nonempty set A of stationary points (A is closed), then for any $\alpha > 0$ there exists an $\epsilon > 0$ such that any closed set on $(f < c + \alpha)$ may be deformed into $(f < c - \alpha) + S(A, \epsilon)$. A has the K-property. The critical points of f are the stationary points.

In what follows in Chapter I we shall without specially saying so suppose that the functionals under consideration possess the K-property.

The principle of Min-Max. *K-homotopy class.* Suppose given a K-class L and a class of K-deformations of K-sets B of L, and that the class L is invariant relative to deformations of that class. We shall call L a homotopy class.

THEOREM 1. *Denote by c the lower limit on L of the upper limits of functions I on sets B of the homotopy class L:*

$$c = \inf[\sup I(x)], \qquad B \in L,\ x \in B.$$

Then c is a K-critical value, i.e., $(I = c)$ contains a nonempty K-critical set of the functional I.

Indeed, if c were a K-ordinary value, then any K-set on $(I \leqq c + \alpha)$ for sufficiently small $\alpha > 0$ could be mapped into the region $(I \leqq c - \alpha)$ by a K-deformation. But for any $\alpha > 0$ there is in $(I \leqq c + \alpha)$ a K-set B of the class L. In view of the definition of L, by means of a K-deformation B maps into a set B_1 of the same class L, lying in $(I \leqq c - \alpha)$, and $\sup I(x)$ on B_1 will be $\leqq c - \alpha$ in spite of the definition of c. The resulting contradiction proves the proposition.

We return to Example 2.

Suppose given a class of closed sets L in R_n, invariant with respect to the deformation operation. If c is the lower limit of the maxima of f on sets of class L, then $(f = c)$ contains stationary points of the function f, at which $df = 0$ (Ljusternik and Šnirel′man [17]).

2. **Merging of critical values.**

Subordinate classes. Suppose given two K-homotopy classes $\mathfrak{B}$ and $\mathfrak{B}_1$, with each set B of the class $\mathfrak{B}$ containing in the set-theoretical sense a set of class $\mathfrak{B}_1$. We shall say that the class $\mathfrak{B}_1$ is subordinate to the class $\mathfrak{B}$. Each of these classes generates corresponding critical values c and c_1 of the functional I.

We have $c_1 \leqq c$. Indeed, for each $\alpha > 0$ there is a set B of the class $\mathfrak{B}$ such that $\sup_{x \in B} I(x) \leqq c + \alpha$. But B has a subset B_1 of the class $\mathfrak{B}_1$, and $\sup_{x \in B_1} I(x) \leqq \sup_{x \in B} I(x) \leqq c + \alpha$, so that

$$c_1 = \inf_{B_1 \in \mathfrak{B}_1} [\sup_{x \in B_1} I(x)] \leqq c + \alpha,$$

from which because of the arbitrariness of $\alpha > 0$ we obtain $c_1 \leqq c$.

The most interesting case is $c = c_1$. Because of the Min-Max principle, $(I = c)$ contains a K-critical set A. For any $\alpha > 0$ there exists a set of class $\mathfrak{B}$ in $(I \leqq c + \alpha)$. Any K-set in $(I \leqq c + \alpha)$ may be deformed into the region

$$(I \leqq c - \alpha) + S(A, \epsilon),$$

where $\epsilon > 0$ and $\alpha = \alpha(\epsilon)$, and $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$. For any $\epsilon > 0$ and $\alpha = \alpha(\epsilon)$ there exists a set B_ϵ of class $\mathfrak{B}$ included in $(f \leqq c - \alpha) + S(A, \epsilon)$.

The critical set A must be such that $B_\epsilon - S(A, \epsilon)$ must not contain sets of class $\mathfrak{B}_1$. In fact, since

$$B_\epsilon \in (I \leqq c\alpha) + S(A, \epsilon),$$

then

$$[B_\epsilon - S(A, \epsilon)] \in (I \leqq c - \alpha).$$

If a set B_1 of class $\mathfrak{B}_1$ lay in $B_\epsilon - S(A, \epsilon)$, then $B_1 \in (I \leqq c - \alpha)$ and $\sup_{x \in B_1} I(x) \leqq c - \alpha < c = c_1$, contrary to the definition of c_1.

The fact that $B_\epsilon - S(A, \epsilon)$ contains no sets of class $\mathfrak{B}_1$, where B_ϵ lies in the class $\mathfrak{B}$, makes it possible in many cases to judge the topological structure of $S(A, \epsilon)$, and thus A as well, from the topological structure of the sets of the classes $\mathfrak{B}$ and $\mathfrak{B}_1$. Here those propositions which make it possible to judge from the properties of two sets the properties of their difference turn out to be particularly important. Therefore in particular invariants of additive character and theorems on the connection between the structure of a set and its complement, of the type of the Alexander duality principle, find their application.

3. Category and other invariants for the finite-dimensional case.

The first in time among the invariants used for this purpose was the invariant called category (Ljusternik [4]). The set M is said to be of category 1, $\operatorname{cat} M = 1$, in the space L, if it is reducible to a point in L by a continuous deformation. The set M is said to be of category m, $\operatorname{cat} M = m$, if it decomposes into m pieces of category 1 but cannot be decomposed into a smaller number of them. Deformation does not reduce the category of a set. In an n-dimensional manifold R_n, for sufficiently small ϵ, $\operatorname{cat} M = \operatorname{cat} S(M,\epsilon)$.

Suppose that $(M)_i$ is the class of sets of category $\geqq i$ in R_n. $(M)_i$ is a homotopy class. Consider a function f in R_n. We denote by

$$c_i = \inf_{M\in(M)_i} \sup_{x\in M} f(x),$$

the critical value of the function f generated by the class $(M)_i$. For $p > 0$, $c_i \leqq c_{i+p}$.

Now suppose that $c_i = c_{i+p} = c$. In the paper cited above it was proved that the critical set A on the level surface $(f = c)$ has category $\geqq p + 1$. Indeed, for sufficiently small ϵ,

$$\operatorname{cat} A = \operatorname{cat} S(A,\epsilon).$$

Moreover, we may determine an $\alpha > 0$, $\alpha = \alpha(\epsilon)$, such that each set in R_n included in $(f \leqq c + \alpha)$ may be transformed into

$$(f \leqq c - \alpha) + S(A,\epsilon)$$

by a continuous deformation.

Suppose that M_{i+p} is a set of class $(M)_{i+p}$, lying in

$$(f \leqq c - \alpha) + S(A,\epsilon).$$

If $\operatorname{cat} A \leqq p$, then for sufficiently small ϵ

$$\operatorname{cat} S(A,\epsilon) \leqq p,$$

$$\operatorname{cat} [M_{i+p} - S(A,\epsilon)] \geqq \operatorname{cat} M_{i+p} - \operatorname{cat} S(A,\epsilon) \geqq i + p - p = i.$$

Thus the difference $M_i = M_{i+p} - S(A,\epsilon)$ is contained in the class $(M)_i$, which is impossible (this difference is contained in $(f \leqq c - \alpha)$, $\sup_{x\in M_i} f(x) \leqq c - \alpha < c = c_i$, contrary to the definition of c_i).

With the aid of the invariant category one may estimate the number of stationary points of a three-times differentiable function f (i.e., the number of points at which $df = 0$) given on a closed manifold R_n. In fact, the critical set on $(f = c)$ is contained in the set of stationary points. Suppose that $\operatorname{cat} R_n = k$. Then one may define the classes $(M)_1, (M)_2, \cdots, (M)_i$ of the sets M_i of category $\geqq i$ $(i = 1, 2, \cdots, k)$. Each class induces a critical value

$$c_i = \inf_{M_i \in (M)_i} \sup_{x \in M_i} f(x).$$

Each level surface $(f = c_i)$ contains a stationary point. If the numbers c_i are distinct, then there are not less than k stationary points. If $c_i = c_{i+p} = c$, which means that the level surfaces $(f = c_i), \cdots, (f = c_{i+p})$ coincide, then on the coincident level surface $(f = c)$ there is a critical set A of category $\geqq p$, so that this set contains an infinite set of stationary points.

We may restrict ourselves to special classes of deformations (the K-deformations, and introduce correspondingly the concept of K-category of a set) with the smallest number of pieces reducible by K-deformations to points into which the set may be decomposed.

The computation of homotopy invariants is quite complicated. For the estimation of category and therefore for the estimation of the number of critical points, L. G. Šnirel′man introduced homology invariants (Šnirel′man [**24**]). These invariants, for example the "rank", turned out to be closely connected with the ring of intersections of the manifold in question. Although these homology invariants gave less exact estimates in certain cases than the homotopy invariants, they were easier in computation, and by means of them it was comparatively simple to compute also the category of certain manifolds, for instance of n-dimensional projective space, the n-dimensional hypertorus and so forth. Among the homology invariants introduced later, the "length of a manifold" defined by S. Frolov and L. Èl′sgol′c as the largest number of cycles of the manifold whose intersection was not zero, turned out to be very convenient in calculation.

It is simple from the logical point of view to generalize the concept of category to the case of n-dimensional spaces, but the calculation of categories then becomes even more complicated. However, it was possible to determine the category of certain sets in the space of closed curves on the sphere and thus to estimate the number of solutions of the variational problem of finding the closed extremals on the sphere or on a surface of genus 0. On such a surface, it turned out, there were always three closed geodesics.

However for other problems, for example for the problem with fixed endpoints, to estimate the category of the family of admissible curves in a direct way did not lie within our grasp.

REMARK. For the estimation of the number of critical points we require both homotopy invariants (category, the invariants introduced by L. E. Èl′sgol′c [29]), and homotopy invariants ("rank", "length", and others). The homotopy invariants yield in general better estimates. Thus Borsuk investigated the connection of category with the fundamental group. It follows from his results that the category of a manifold F_n whose homology structure coincides with the structure of the n-dimensional sphere S_n, and whose fundamental group is not trivial, is not less than 3, so that for any function on F_n the number of critical points $\geqq 3$, while for a function on the sphere it may be equal to 2. Meanwhile, from the point of view of homology F_n does not differ from S_n. That such manifolds exist was shown by Poincaré, starting with dimension $n = 3$.

4. Dimension of critical sets. The n-dimensional case.

We shall consider in the n-dimensional manifold R_n classes of k-dimensional cycles (z_k) such that along with the cycle z_k the class (z_k) contains all the cycles homologous to qz_k. Here, as everywhere in what follows, q is an arbitrary positive number less than p in the case that the homologies are taken modulo a prime p, and q is an arbitrary nonzero whole number if the homologies are taken modulo 0.

If the homologies are taken with the real numbers for coefficients, then q denotes an arbitrary real number distinct from zero, and so forth.

We shall say that the class (z_k) of k-dimensional cycles z_k is subordinated to the class (z_{k+l}) of $(k+l)$-dimensional cycles z_{k+l} by means of the class (z_{k+l}) of $(n-l)$-dimensional cycles, if the intersection of the cycle z_{n-l} of (z_{n-l}) with the cycle z_{k+l} of (z_{k+l}) is a cycle z_k of the class (z_k). Since the cycle classes (z_k), (z_{k+l}) are invariant with respect to the deformation operation, the classes (z_k), (z_{k+l}) are at the same time homotopy classes subordinate one to the other.

THEOREM 2. *If the critical values* c_k, c_{k+l} *of the function* f *generated by the classes* (z_k), (z_{k+l}) *coincide,* $c_k = c_{k+l} = c$, *then the level surface* $f = c$ *contains a critical set* A *whose dimension is not less than* l.

This theorem may be sharpened.

Theorem 3. *If* $c_{k+l} = c_k = c$ *and* A *is a critical set on* $(f = c)$, *then* A *contains an* l*-dimensional cycle* z_l *not homologous to zero, and such that* $z_l \times z_{n-l} \neq 0$, $z_{n-l} \in (z_{n-l})$.

In fact, suppose the contrary. For any l-dimensional cycle z_l lying in A, $z_l \times z_{n-l} = 0$, where z_{n-l} is a cycle of the class (z_{n-l}). The theorem on removing cycles of L. S. Pontrjagin [23] states that if for any l-dimensional cycle z_l lying in A, $z_l \times z_{n-l} = 0$, then there exists a cycle $\bar{z}_{n-l}$ homologous to z_{n-l} and lying entirely outside A. Since A is closed, there exists an $\epsilon > 0$ such that $\bar{z}_{n-l}$ lies outside $S(A, \epsilon)$. There exists an $\alpha > 0$ such that every closed set in $(f \leqq c + \alpha)$ may be deformed into $(f \leqq c - \alpha) + S(A, \epsilon)$. Since for any $f \leqq c + \alpha$ there exists, because of the definition of c, a cycle $z_{k+l} \in (z_{k+l})$ lying in the region $(f \leqq c + \alpha)$, this means that there exists a cycle $\bar{z}_{k+l} \sim z_{k+l}$ $(\bar{z}_{k+l} \in (z_{k+l}))$ lying in $(f \leqq c - \alpha) + S(A, \epsilon)$. Write $z_k = \bar{z}_{k+l} \times \bar{z}_{n-l}$. By definition z_k lies in the class (z_k). Since $\bar{z}_{k+l} \in (f \leqq c - \alpha) + S(A, \epsilon)$, and $\bar{z}_{n-l}$ lies outside $S(A, \epsilon)$, therefore z_k, situated in the set-theoretical intersection of $\bar{z}_{k+l}$ and z_{n-l}, lies in $(f \leqq c - \alpha)$, and the maximum of f on z_k does not exceed $c - \alpha < c = c_k$, which contradicts the definition of c_k. The theorem is proved.

We may consider a lattice whose elements are homology classes of the manifold, with the class (z_{k+l}) following the class (z_k) if the elements of the class (z_k) are obtained as intersections of elements of the class (z_{k+l}) with cycles z_l of the class (z_l). This theorem indicates the order of appearance of critical values of the function f. If the class (z_{k+l}) follows the class (z_k), the critical values c_k, c_{k+l} generated by them satisfy the inequality $c_k < c_{k+l}$, or in the case of coincidence $c_k = c_{k+l}$ there appears a continual set of critical points.

The largest number of homology classes of the manifold following one another coincides with the invariant "length of a manifold" (Frolov and Èl′sgol′c [3]).

5. Transition to compacta.

On removal of ∇*-cycles.* We shall carry the results obtained above to the case of an arbitrary compactum. In the proof of the theorem we used the Pontrjagin theorem on separation of a cycle. The theorem was generalized by P. S. Aleksandrov to the case of compacta (more generally to locally bicompact spaces) M.

Theorem 4 (Aleksandrov [2a]). *Suppose that in the locally bicompact space* M *there is an* n*-dimensional* ∇*-cycle* Z^n *and a closed set* A *such that for any* n*-dimensional* ∇*-cycle of* A *the intersection index* $Z^n \times z_n = 0$.

Then the cycle Z^n *may be removed from* A, *i.e., there exists a* ∇*-cycle* $\overline{Z}_n$ ∇*-homologous to* A^n *and lying outside* A, *i.e., the support of* $\overline{Z}_n$(1) *lies outside* A (*and moreover, for sufficiently small* $\epsilon > 0$, *outside* $S(A, \epsilon)$.

As a consequence of the theorem we prove the following:

Suppose that $Z^{n+m} \overset{\sim}{\nabla} Z^n \times Z^m$, where Z^n, Z^m, Z^{n+m} are n, m, and $(n+m)$-dimensional ∇-cycles, and z_{n+m} is an $(n+m)$-dimensional Δ-cycle such that $z_{n+m} \times Z^{n+m} \neq 0$. Then the set-theoretical intersection of the support of Z^n and z_{n+m} contains an m-dimensional Δ-cycle z_m such that $Z^m \times z_m \neq 0$.

In fact, suppose that A is the set-theoretical intersection of z_{n+m} and the support of Z^n. If for any m-dimensional Δ-cycle z_m of A we have $z_m \times Z^m = 0$, then there exists a ∇-cycle $\overline{Z}^m \overset{\sim}{\nabla} Z^m$ whose support lies outside A. Suppose that $\overline{Z}^{n+m} = Z^n \times \overline{Z}^m$. We have $\overline{Z}^{n+m} \overset{\sim}{\nabla} Z^{n+m}$. The support of $\overline{Z}^{n+m}$ lies in the set-theoretical intersection of the supports of Z^n, $\overline{Z}^m$, and z_{n+m}. But this last intersection is empty. In fact, the support of $\overline{Z}^m$ lies outside $S(A, \epsilon)$, i.e., outside the set-theoretical intersection of the support of Z^n and z_{n+m}. But it therefore follows that $(Z^n \times \overline{Z}^m) \times z_{n+m} = 0$. Meanwhile

$$(Z^n \times \overline{Z}^m) \times z_{n+m} = \overline{Z}^{n+m} \times z_{n+m} = Z^{n+m} \times z_{n+m} \neq 0.$$

The resulting contradiction proves the assertion.

Čech gave a definition of the product of the upper ∇-cycle Z^n and the Δ-cycle z_{n+m} of respective dimensions n and $n+m$. This product is an m-dimensional Δ-cycle z_m, lying in the set-theoretical intersection of z_{n+m} and the support of Z^n, while if

$$(Z^n \times Z^m) \times z_{n+m} \neq 0,$$

then

$$Z^m \times z_m \neq 0.$$

The existence of a cycle satisfying these conditions follows from the consequence presented above of the theorem on removal of cycles.

K-classes of Δ*-cycles*. A Δ-cycle which is a K-set will be called a K-cycle. A K-class (z_k) will be a family of Δ-cycles such that along with z_n in (z_n) it also contains every Δ-cycle homologous to qz_n, where $q \neq 0$

---

(1) The support of the ∇-cycle U is a closed set B with the following properties: (1) for any neighborhood of B, U may be defined by a sequence of functions equal to zero outside that neighborhood; (2) B is irreducible with respect to property (1).

for homologies modulo 0, $0 < p < q$ for homologies modulo a prime p, and more generally, q is any nonzero element of the group of coefficients of the homology.

A K-class of n-dimensional Δ-cycles (z_n) is subordinated to the K-class of $(n+m)$-dimensional Δ-cycles (z_{n+m}) by means of ∇-cycles of the class (Z^m) if in the set-theoretical intersection of the support of any ∇-cycle $Z^m \in (Z^m)$ and any cycle z_{n+m} of (z_{n+m}) contains a Δ-cycle z_n of class (z_n).

Suppose that (Z^n), (Z^m), and (Z^{n+m}) are homology classes of n, m, and $(n+m)$-dimensional ∇-cycles such that $Z^n \in (Z^n)$, $Z^m \in (Z^m)$, $(Z^n \times Z^m) \in (Z^{n+m})$.

Suppose further that (z_n) and (z_{n+m}) are the classes of all n- and $(n+m)$-dimensional Δ-cycles such that for z_n, z_{n+m} of (z_n), (z_{n+m}) and Z^n, Z^{n+m} of (Z^n), (Z^{n+m}),

$$Z^n \times z_n \neq 0,$$

$$Z^{n+m} \times z_{n+m} \neq 0.$$

From the consequence given above of the theorem on removal of cycles it follows that the class (z_n) is subordinated to the class (z_{n+m}) by means of the class (Z^m).

If we are given in M a K-class (z_n), then it is also a K-homotopy class. If a function $f(a)$ is given in M and

$$c = \inf_{a \in (z_n)} f(a),$$

then c is a critical value, i.e., $(f = c)$ contains a nonempty K-critical set (Theorem 1). The critical value c will be said to be generated by the class (z_n).

Now we shall prove a generalization of Theorem 3 to the case of a compactum M.

THEOREM 5 (FUNDAMENTAL). *Suppose that $f(a)$ is a function given on M, (z_n) and (z_{n+m}) classes of n- and $(n+m)$-dimensional Δ-cycles, while the class (z_n) is subordinated to (z_{n+m}) by means of the ∇-class (Z^m). Suppose further that c_n and c_{n+1} are the K-critical values generated by the classes (z_n) and (z_{n+m}):*

$$c_n = \inf_{z_n \in (z_n)} \sup_{a \in z_n} f(a);$$

$$c_{n+m} = \inf_{z_{n+m} \in (z_{n+m})} \sup_{a \in z_{n+m}} f(a).$$

If $c_n = c_{n+m} = c$, *then the* K-*critical set* A *on* $(f = c)$ *contains an* m-*dimensional* Δ-*cycle* z_m *not homologous to zero, for which* $z_m \times Z^m \neq 0$.

The critical values c_n and c_{n+m} are connected by the relation $c_n \leqq c_{n+m}$. The theorem states that if this inequality turns into an equation, then the corresponding merging level surface contains a critical set of dimension not less than m.

Thus, suppose that $c_n = c_{n+m} = c$ and suppose that A is a K-critical set on the level surface $(f = c)$. If for any m-dimensional Δ-cycle z_m, contrary to the assertion of the theorem, $z_m \times Z^m = 0$, then the cycle Z^m may be removed from A, i.e., there exists some cycle $\overline{Z}^m$ of the same class (Z^m) whose support lies outside $S(A, \epsilon)$, where ϵ is some positive number. There exists a number $\alpha > 0$ such that any K-set of $(f \leqq c + \alpha)$ may be brought into $(f \leqq c - \alpha) + S(A, \epsilon)$ by a K-deformation. By definition, $c = c_{n+m}$ and in $(f \leqq c + \alpha)$ there exists a cycle z_{n+m} of the class (z_{n+m}). By a K-deformation it may be carried into a cycle $\bar{z}_{n+m}$ of the same class (z_{n+m}), lying in $(f \leqq c - \alpha) + S(A, \epsilon)$. In the set-theoretical intersection of the support of $\overline{Z}^m$ of (Z^m) and $\bar{z}_{n+m}$ of (z_{n+m}) there is a cycle z_n of the class (z_n). This follows from the definition of subordination of the classes (z_n) and (z_{n+m}) by means of the class (Z^m).

But since $\bar{z}_{n+m}$ lies in $(f \leqq c - \alpha) + S(A, \epsilon)$, and the support of $\overline{Z}^m$ outside $S(A, \epsilon)$, therefore z_n lies in $(f \leqq c - \alpha)$:

$$\sup_{a \in z_n} f(a) \leqq c - \alpha = c_n - \alpha < c_n,$$

in contradiction to the definition of c_n. This contradiction proves the theorem.

Illustrative example. *Problem with movable endpoints on the plane.* Consider on the plane a closed curve p with everywhere continuous curvature and a point A within p.

The family of rectifiable plane arcs with origin at A and endpoint on p forms a space T which may be metrized, taking the distance in the sense of Fréchet. Suppose that the length of the curve q of T is equal to $I(q)$.

On the curve q we introduce a parameter t such that when a point moving along the curve q from A describes a path of length s the endpoint a_q^t of this path corresponds to the value $t = s/I(q)$ of the parameter. For $t = 1$, a_q^1 is the endpoint of the curve q lying on the curve p, and $a_q^0 = A$. By a K-set in T we shall mean a closed set of curves q of T on which the points a_q^t depend continuously on the curve q and the parameter t. For example, the closed set of curves of bounded curvature or the closed set of polygons with a bounded number of sides will form a K-set.

Every K-set M of T may be carried by a continuous deformation into a set of rectilinear segments. Suppose that $r \in M$ and a_r^1 is the endpoint of r lying on p. Denote by D_r the rectilinear segment with origin at A and endpoint a_r^1, the parameter t on D_r being taken according to the rule chosen for $q \in T$. Join each point a_r^t of the curve r to the point $a_{D_r}^t$ of the corresponding segment D_r by a rectilinear path $\overline{a_r^t a_{D_r}^t}$. The deformation D_r consists of moving each point a_r^t of the curve r along the segment $\overline{a_r^t a_{D_r}^t}$ in the direction of $a_{D_r}^t$ with a velocity proportional to the length of the latter, so that in a unit time that point coincides with $a_{D_r}^t$ and the curve r goes into the segment $Dr = D_r$, and the set M into the set DM of such rectilinear segments.

Suppose that Dr is not a normal to p. Then one of the two rays tangent to p issuing from the point $c = a_{D_r}^1$ which we shall denote by n, forms an acute angle α with Dr. We shall transfer the segment $Dr = Ac$, leaving invariant its origin A, and transferring the endpoint c along the curve p in the direction of the tangent ray n with a velocity proportional to $\cos\alpha$, namely the cosine of the angle between the segment and this direction of the tangent. When this is done the length $I(\overline{Ac})$ of the segment Ac will decrease. The velocity will tend to zero as $\overline{Ac}$ approaches the normal k of the curve p drawn from A. The segments Ad, normal to p, remain unshifted. Carrying out such a shift in the course of some interval of time, we decrease the lengths of all the segments $\overline{Ac}$ which are not normal to p. Such a transformation is a continuous deformation D_1. The successive application of the operations D and D_1 is a deformation D_1D, leaving invariant the normals to the curve p and decreasing the lengths of all the other curves of M.

Suppose that M_0 is the set of normals to p drawn from A. For each $\epsilon > 0$ we may define an $\epsilon_1 > 0$ such that curves q distant from some normal in M_0 by not more than ϵ_1 recede from that normal by no more than ϵ.

Further, we may determine an α such that curves of M which differ from normals to p by more than ϵ decrease in length under the operation D_1D by no less than 2α. Further, we lay on α the requirement that all the normals of M_0 whose lengths are included between $c - \alpha$ and $c + \alpha$ lie in $S(M_l^0, \epsilon)$, where M_l^0 is the family of normals of length l, $M_l^0 \in (I = l)$. Therefore it follows that the functional I has the K-property and the critical set on $(I = l)$ lies in the set of normals to p of length l.

Indeed, suppose that T' is the set of arcs of T lying in $(c - \alpha < I < c + \alpha)$.

After the deformation D_1D this goes into a set lying in $(I \leqq c - \alpha) + S(M_l^0, \epsilon)$.

Now we choose a point B inside p and distinct from A. Suppose that q_1 and q_2 are two curves of T not passing through B, such that $\rho(q_1, q_2) < \alpha$, where α is a positive number sufficiently small that the points $\alpha_{q_1}^1$ and $\alpha_{q_2}^1$ on p are distant from one another by less than ϵ.

For a sufficiently small ϵ the points $a_{q_1}^1 = d_1$ and $a_{q_2}^1 = d_2$ bound two arcs on p, one of which is small than $I(p)/3$ in length. We denote it by $\overset{\smile}{d_1d_2}$. We denote by $f_1^\alpha(q_1, q_2, B)$ the function equal to the index modulo 2 relative to the point B of the curve $q_1 + \overset{\smile}{d_1d_2} - q_2$.

The system of functions $f_1^\alpha(q_1, q_2, B)$ defines on T a one-dimensional upper cycle Z_B^1. (Z^1) is the homology class for Z_B^1.

The family of rectilinear segments joining A to the points of p forms on T a one-dimensional Δ-cycle z_1. The intersection index $z_1 \times Z^1 = 1$.

We denote by (z_1) the family of all Δ-cycles of the class of z_1, i.e., homologous to z_1 (homologies are taken modulo 2), and by (z_0) the family of all the remaining elements of T, i.e., if desired, the family of zero-dimensional cycles not homologous to zero. Evidently if a function $I(q)$ is given on T and

$$c_1 = \inf_{z_i \in (z_i)} \sup_{q \in z_i} I(q), \quad i = 0, 1,$$

then c_0 is simply the lower limit of $I(q)$. Evidently $c_0 \leqq c_1$. Since c_0 and c_1 are K-critical values (see §1), i.e., $(I = c_0)$ and $(I = c_1)$ must contain nonempty K-critical sets, and therefore nonempty sets of normals to the curve p of lengths c_0 and c_1.

The class (z_0) is subordinated to the class $(\bar{z}_1)$ by means of Z^1.

If $c_0 = c_1 = c$, then $(I = c)$ contains a one-dimensional cycle $\bar{z}_1$ consisting of normals dropped from A to p, of length l, $\bar{z}_1 \times Z_1 = 1$ (in view of the fundamental theorem), the $\bar{z}_1$, are nonhomologous to zero mod 2. A family of segments joining A to p and not containing all the segments may be continuously deformed into a single segment and therefore does not contain cycles nonhomologous in T. Therefore if a family of normals of length c contains a one-dimensional cycle $\bar{z}_1$, nonhomologous to zero, then all the segments joining A to points of p are normals dropped from A to p and of length c, i.e., p is a circumference with center at A and radius C. These results may be obtained, of course, by elementary considerations. We have presented them here only as a type of problem arising in the solution of more complex problems.

CHAPTER II

Application to nonlinear integral equations

6. **Homogeneous operators.** As is well known, the Fredholm integral equation with positive symmetric kernel has a countable number of positive eigenvalues, tending to zero. In the formulation of this theorem one cannot drop the requirement of positivity, let alone symmetry. On the other hand one may drop the condition of linearity, replacing it by a more general condition of odd homogeneity or simply oddness. Odd homogeneity of an equation was considered in our paper [**9**]. Here we give a more complete theory of these equations as an illustration of the general methods. An application of the theory of category to these equations was given in the work of Citlanadze [**26**].

For a complete continuous symmetric positive operator L in Hilbert space there exists a countable number of normalized eigenelements

$$La_n = \lambda_n a_n,$$

where

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq \cdots, \qquad \lambda_n > 0, \ \lambda_n \to 0.$$

If $\lambda_n = \lambda_{n+p} = \lambda$, the multiple eigenvalue λ corresponds to a p-dimensional sphere of normalized eigenelements.

We shall now consider nonlinear operators L. An operator L is called completely continuous if it maps any bounded set in real Hilbert space H into a compact set and satisfies a Lipschitz condition:

$$\| Lb' - Lb \| \leqq M \| b' - b \|,$$

where M is a constant.

Suppose that a functional f is given in H. The differential $df(a;h)$ of f at the point a for the increment h is the quantity

$$df(a;h) = df(a + th)/dt|_{t=0}, \tag{1}$$

if the right side exists and is a linear functional of h. In this case there exists an element $g \in H$ such that $df(a;h) = (g,h)$.

Here the element g depends on a: $g = La$,

$$df(a;h) = (La,h). \tag{2}$$

The operator La generated by differentiation of the functional f is naturally called symmetric. In the case of a linear symmetric operator the latter is generated by the differentiation of a quadratic functional.

A positive operator is an operator such that $(La,a) > 0$ when $a \neq 0$.

Now suppose that $f(a)$ is a homogeneous even functional of order k, i.e.,

$$f(-a) = f(a),$$

and, if $\lambda > 0$,

$$f(\lambda a) = \lambda^k f(a).$$

In this case, if La is an operator generated by the differentiation of $f(a)$, then

$$L(-a) = -La,$$

and if $\lambda > 0$, $L(\lambda a) = \lambda^{k-1}La$. La is an odd homogeneous operator of order $k-1$. We shall suppose $k > 1$.

For homogeneous functions of order k there is an analogue of Euler's theorem:

$$(La,a) = kf(a). \tag{3}$$

In fact, $df(a;h) = (La,h)$. Therefore

$$\begin{aligned}(La,a) &= df(a;a) = df(a+ta)/dt|_{t=0}\\ &= [d(1+t)^k/dt]_{t=0}f(a) = kf(a).\end{aligned}$$

We shall consider normalized eigenelements a of the operator L:

$$La = \lambda a, \quad \|a\| = 1. \tag{4}$$

and the corresponding eigenvalues λ.

If La is generated by differentiation of the homogeneous function $f(a)$ of order k, it follows from (3) and (4) that

$$kf(a) = (La,a) = \lambda(a,a) = \lambda.$$

We shall write

$$f_1(a) = kf(a) = (La,a).$$

We then have:

THEOREM 1. *Suppose that La is a completely continuous, symmetric, odd-homogeneous positive operator in Hilbert space H. There exists a sequence of positive eigenvalues λ_n of the operator L:*

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq \cdots, \qquad \lambda_n > 0,$$

tending to zero. If $\lambda_n = \lambda_{n+p} = \lambda$, *then to the multiple eigenvalue* λ *there corresponds a p-dimensional system of normalized eigenvalues.*

The last part of this theorem will be made more precise in what follows. Below in this section and in the following §7, we shall give the proof of this theorem.

ORTHOGONAL TRAJECTORIES

We shall denote by $[b(t)]_0^\alpha$ the strongly continuous image in H of the segment $[0 \leqq t \leqq \alpha]$. Consider the equation

$$db(t)/dt = B(b(t)), \tag{5}$$

where B is an operator in H satisfying in $S(b_0, h)$ the Lipschitz condition

$$\| B(b') - B(b'') \| < N \| b' - b'' \|,$$

and N is a positive constant.

We now have the following existence theorem:

There exists a unique solution $[b(t)]_0^\alpha$, of the equation (5) which lies in $S(b_0, h)$ with any initial condition $b(0) = b_1$, where

$$b_1 \in S(b_0, h/2),$$

with α defined by h and N. Here $b(t)$ for the segment $[0, \alpha]$ depends in a uniformly continuous way on the initial values $b(0)$.

The proof of this theorem is an immediate generalization of the proof of the analogous theorem for the n-dimensional case by the method of Picard. Consider now the equation

$$db(t)/dt = Lb(t) - f_1(b(t))\, b(0). \tag{6}$$

The right side of this equation satisfies a Lipschitz condition in the sphere $\|b\| < 2$. In fact, suppose that $\|b'\| < 2$, $\|b''\| < 2$. We have

$$\begin{aligned}
&\| Lb' - f_1(b')\, b' - Lb'' + f_1(b'')\, b'' \| \\
&\quad \leqq \| Lb' - Lb'' \| + | f_1(b') - f_1(b'') |\, \| b' \| + | f(b'') | \cdot \| b' - b'' \| \\
&\quad \leqq \| Lb' - Lb'' \| + | (Lb', b') - (Lb'', b'') | \cdot \| b' \| \\
&\qquad\qquad + \| Lb'' \| \cdot \| b'' \| \cdot \| b' - b'' \| \\
&\quad \leqq \| Lb' - Lb'' \| + \| b' \| \cdot \| Lb' - Lb'' \| + \| Lb'' \| \cdot \| b' - b'' \| \\
&\qquad\qquad + \| Lb'' \| \cdot \| b'' \| \cdot \| b' - b'' \| \\
&\quad = \| Lb' - Lb'' \| (1 + \| b' \|) + \| b' - b'' \| \cdot \| Lb'' \| (1 + \| b'' \|).
\end{aligned}$$

If we denote by R the maximum of $\|Lb\|$ in the sphere $\|b\| < 2$ and recall that

$$\|b'\| \leqq 2, \ \|b''\| < 2, \quad \text{and} \quad \|Lb' - Lb''\| \leqq M\|b' - b''\|,$$

then the preceding implies that

$$\|(Lb' - f_1(b')\,b') - (Lb'' - f_1(b'')\,b'')\| \leqq (3M + 2R)\|b' - b''\|,$$

i.e., the right side of our equation satisfies a Lipschitz condition. This means now that there exist integral curves $[b(t)]_0^\alpha$ of this equation in the sphere $\|b\| < 2$ with arbitrary initial values $b(0)$, where $\|b(0)\| = 1$, and these solutions depend uniformly on the initial values on the segment $[0, \alpha]$.

Suppose that $\Psi(t) = 1 - \|b(t)\| = 1 - (b(t), b(t))$. We have

$$\Psi'(t) = -2(b(t), db(t)/dt) = -2[Lb(t), b(t)] - 2f_1(b(t))\|b(t)\|$$
$$= -2f_1(b(t))\,[1 - \|b(t)\|] = -2f_1(b(t))\,\Psi(t).$$

If for some t we have $\|b(t)\| = 1$, then $\Psi(t) = 0$. In the sphere

$$\|b\| < 2, \ \|f_1(b)\| = \|Lb, b\| \leqq \|Lb\| \cdot \|b\| \leqq 2N$$

and therefore the solution of the equation

$$\Psi'(t) = -2f_1(b(t))\,\Psi(t)$$

under the initial conditions $\Psi(0) = 0$ is $\Psi(t) \equiv 0$. Therefore, if the point $b(0)$ lies in S, $\|b(0)\| = 1$, $\Psi(t) \equiv 0$, $\|b(t)\| = 1$, and the entire curve $b(t)$ lies in S.

Since not only $[b(t)]$ but also $[-b(t)]$ satisfies equation (6), we may say that $[b(t)]$ is a curve on P.

Further

$$df_1[b(t)]/dt = (Lb(t), db(t)/dt) = (Lb(t), Lb(t)) - f_1(b(t))(Lb(t), b(t))$$
$$= (Lb(t), Lb(t)) - [Lb(t), b(t)]^2.$$

Because $(b(t), b(t)) = 1$, it follows from the Schwartz inequality that the right side of this equation is always nonnegative and is only equal to zero when $b(t)$ is an eigenelement. The curves $[b(t)]$ will be called orthogonal trajectories for the level surface $f_1 = c$, with the parameter t increasing on them with increasing f_1. If $b(0)$ is an eigenelement, then the curve $[b(t)]$ degenerates into the point $b(0)$.

7. Compact parts of projective space P.

Compact sets on P. The condition that the set A is compact in H is the existence of a sequence of numbers α_n such that $\alpha_n \to 0$ and for any $a \in A$, $\| R_n a \| \leqq \alpha_n$; if $a = (a_1, a_2, \cdots, a_n, a_{n+1}, \cdots)$, where the a_i are the components of a in H, then

$$R_n a = (0, 0, \cdots, 0, a_{n+1}, a_{n+2}, \cdots).$$

Since La is a completely continuous operator, then for any $a \in S$

$$\| R_n La \| \leqq \alpha_n, \tag{7}$$

where α_n is a fixed sequence of positive numbers and $\alpha_n \to 0$.

Consider on S or P the region $(f_1 \geqq c)$, where $c > 0$. If a is an eigenelement in this region, $La = \lambda a$, $\lambda = f_1(a) \geqq c$ and from (7)

$$\| R_n a \| = (1/\lambda) \| R_n La \| \leqq (1/c) \| R_n La \| \leqq \alpha_n / c. \tag{8}$$

We select from $\{\alpha_n\}$ a subsequence $\{\alpha_{n_k}\}$ such that $\sum \alpha_{nk}$ converges (for example $\alpha_{n_k} \leqq 1/2^k$). Denote by P_c the compact set in P or S defined by the system of inequalities

$$f_1 \geqq c, \ \| R_{n_k} \| \leqq 2\alpha_{n_k}/c, \tag{9}$$

where we restrict ourselves to those inequalities in which $\alpha_{n_k} < c/2$, the others being fulfilled automatically, since we always have

$$\| R_n a \| \leqq \| a \| = 1.$$

From (8) all the eigenelements lying in $(f_1 \geqq c)$ belong to P_c.

Suppose that n_i is the smallest of the numbers n_k in (9) such that $\alpha_{n_k} < c/2$.

We put $n \geqq n_i$, so that

$$\| R_n La \| \leqq \alpha_n \leqq \alpha_{n_i} < c/2. \tag{10}$$

For each a, where $a = (a_1, a_2, \cdots, a_n, a_{n+1}, \cdots)$, we write

$$A_n a = (a_1, a_2, \cdots, a_n, 0, 0, \cdots).$$

Evidently

$$a = A_n a + R_n a.$$

If

$$a \in (f_1 \geqq c),$$

then

$$(La, a) = f_1(a) \geqq c.$$

But

$$(La, a) = (A_n La, A_n a) + (R_n La, R_n a),$$
$$(R_n La, R_n a) \leqq \| R_n La \| \, \| R_n a \| \leqq (c/2) \cdot 1 = c/2,$$

so that

$$(11) \qquad (A_n La, A_n a) = (La, a) - (R_n La, R_n a) > c - c/2 = c/2,$$

from which

$$\| A_n(a) \| \neq 0,$$

if

$$a \in (f_1 > c).$$

THE DEFORMATION D

We now choose $m \geqq n_i$, where n_i is the quantity just defined.
We define a deformation D_m:

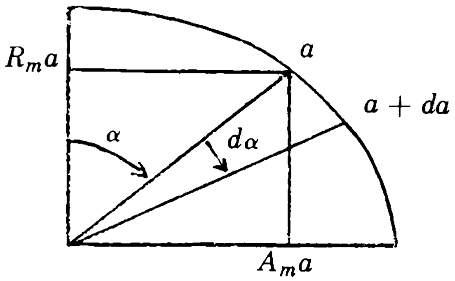

FIGURE 1.

(1) If $\| R_m a \| \leqq 2\alpha_m/c$, then $D_m a = a$.

(2) Suppose that $\| R_m a \| \geqq 2\alpha_m/c > 0$, while the first two terms differ from zero. We verify that in $(f_1 > c)$

$$\| A_m(a) \| \neq 0.$$

We define the deformation $D_m a$ of the element a in the following way. Through Θ and both the orthogonal nonnull elements $A_m a$ and $R_m a$ we pass a plane and in it a circumference q of radius 1 with center at Θ. The element a lies on q, as in Figure 1. The deformation D_m for the element a will consist in moving the element a along the circumference q in the direction of decreasing norm of the component $R_m a$, so long as it does not become equal to $2\alpha_m/c$.

If the element a moves along q, describing the arc $d\alpha > 0$ in the direction of decreasing $\| R_m a \|$, then

$$da = \left(A_m(a) \frac{\| R_m a \|}{\| A_m a \|} - R_m a \frac{\| A_m a \|}{\| R_m a \|} \right) d\alpha. \tag{12}$$

If β is a portion of the arc q included between a and $A_m(a)/\| A_m a \|$, then $\sin \beta = \| R_m a \|$.

Since $\beta < 2 \sin \beta = 2\| R_m a \|$, and the shift of the element a under the deformation D_m does not exceed β, then

$$\| D_m a - a \| \leqq \| R_m a \|.$$

The operation D_m defined in this way is a deformation, with the following properties:

$$\begin{aligned} &(1) \quad \text{if } b = D_m a, \text{ then } \| R_m b \| \leqq 2\alpha_m / c; \\ &(2) \quad \| b - a \| \leqq \| R_m a \|; \\ &(3) \quad f_1(b) = f_1(D_m a) \geqq f_1(a). \end{aligned} \tag{13}$$

Indeed, if a receives a differential increment da defined by formula (12) for $d\alpha > 0$, then

$$\begin{aligned} df_1(a, da) &= (La, da) = (A_m La, A_m da) + (R_m La, R_m da) \\ &= \left[(A_m La, A_m a) \frac{\| R_m a \|}{\| A_m a \|} - (R_m La, R_m a) \frac{\| A_m a \|}{\| R_m a \|} \right] d\alpha. \end{aligned}$$

Since [see (11) and (9)]

$$(A_m La, A_m a) > c/2, \quad \| A_m a \| \leqq \| a \| = 1,$$

$$(R_m La, R_m a) \leqq \| R_m La \| \cdot \| R_m a \| \leqq \alpha_m \| R_m a \|,$$

therefore

$$df_1(a, da) \geqq ((c/2) \| R_m a \| - \alpha_m)\, d\alpha,$$

so that if $\| R_m a \| \geqq 2\alpha_m / c$,

$$df_1(a, da) \geqq 0.$$

Thus under the described shift along the circumference q the function f_1 increases as long as $\| R_m a \|$ remains larger than $2\alpha_m / c$. Therefore (13) follows from the definition of the deformation D_m.

Now we apply successively the operations $D_{n_{i+1}}, D_{n_{i+2}}, \cdots, D_{n_k}$ to the element a of $(f_1 \leqq c)$ (for all $k > i$). We write

$$a_k = D_{n_k} D_{n_{k-1}} \cdots D_{n_{i+1}} a = D_{n_k} a_{k-1}.$$

Since

$$a_{k-1} = D_{n_{k-1}} a_{k-2} \quad \text{and} \quad \| R_{n_{k-1}} a_{k-1} \| \leqq 2\alpha_{n_{k-1}}/c,$$

$$\| R_{n_k} a \| \leqq \| R_{n_{k-1}} a \| \leqq 2\alpha_{n_{k-1}}/c,$$

then, because of the property of D_m mentioned above, the shift

$$\| a_k - a_{k-1} \| \leqq 2\alpha_{n_{k-1}}/c.$$

The series $\sum_{k=i+1}^{\infty} (2\alpha_{n_k}/c)$ converges, and along with it the series $\sum_{k=i+1}^{\infty} (a_k - a_{k-1})$ as well and in fact absolutely.

The sequence

$$a_k = D_{n_k} D_{n_{k-1}} \cdots D_{n_{i+1}} a$$

converges uniformly in the entire region $a \in (f_1 \geqq c)$.

Suppose that $a_k \to a_\omega$.

The operation D.

$$Da = a = \lim_{k \to \infty} (D_{n_k} D_{n_{k-1}} \cdots D_{n_{i+1}} a)$$

is a continuous deformation of the region $(f_1 \geqq c)$. Since for the element

$$a_\omega = Da, \quad \| R_{n_k} Da \| \leqq 2\alpha_{n_k}/c,$$

therefore Da lies in the compactum P_c. Further

$$f_1(Da) \geqq f_1(a) \geqq c. \tag{14}$$

Thus the operation D converts all of $(f_1 \geqq c)$ into the compact portion

$$P_c \subset (f_1 \geqq c)$$

of this region, P_c remaining invariant under this deformation.

The operation D^n. In the following section we shall need an operation related to D_n. Suppose that z_k denotes projective k-dimensional space, in fact the k-dimensional unit sphere $R_{k+2} a = \Theta$, $\| a \| = 1$ with identification of opposite elements.

Further, suppose that l_i is the ith unit coordinate vector:

$$A_{i-1} l_i = \Theta, \quad A_i l_i = l_i, \quad R_{i-1} l_i = l_i.$$

The pair $(l_i, -l_i)$ lies in z_n for $i = 1, 2, \cdots, n+1$. Further,

$$f(l_i) = (Ll_i, l_i) = (R_{i-1} L l_i, R_{i-1} l_i) \leqq \| R_{i-1} l_i \| \quad \| R_{i-1} L l_i \|$$

$$\leqq \| R_{i-1} L l_i \| \leqq \alpha_i.$$

If $c \geqq \alpha_i$, then l_i does not lie in $(f_1 \geqq c)$, and thus also z_i does not lie in $(f_1 \geqq c)$.

Further, for a fixed c in the region $(f_1 \geqq c)$, and in particular in P_c, we have $\| R_n a \| < d < 1$ for sufficiently large n. We define an operation D^n converting the region $(f_1 \geqq c)$ into the part z_n. Indeed, if $R_n a = \theta$, so that $a \in z_{n-1}$, we leave a unaltered. If $R_n a \neq 0$, we pass a plane through the elements θ, $A_n a$, $R_n a$ and in it a circumference of unit radius with center at θ, on which a lies. The operation D^n will consist in shifting a along that circumference in the direction of decreasing $\| R_{n+1} a \|$ so long as this projection does not become equal to zero, i.e., so long as a does not fall into z_{n-1}: $D_n a \in z_{n-1} \cdot D^n$ is a deformation carrying $(f_1 \geqq c)$ and, in particular, P_c, into z_{n-1}. For sufficiently small $c' > 0$, z_{n-1} lies in $P_{c'}$ and the operation D^n carries P_c into its subset z_n inside $P_{c'}$.

The operation G. We define the following operation G on the compact set P_c.

(1) if a is an eigenelement lying in P_c

$$Ga = a;$$

(2) if $a \in P_c$ is not an eigenelement, then we may pass a normal trajectory $[b(t)]_0^{t_1}$ through a which satisfies equation (6) and the initial condition $b(0) = a$.

As was proved above, for $t > 0$, $f_1(b(t)) > f_1(b(0))$.

For fixed $t_0 > 0$, we denote by a_t for $0 \leqq t \leqq t_0$, $a_t = D[b_a(t)]$. Evidently $a_0 = Db(0) = Db(0) = Da = a$ (since $a \in P_c$ and the operation D does not alter $\overline{P}_c$). For a a noneigenelement we put

$$Ga = a_{t_0} = D[b_a(t_0)].$$

We have [see (14)]

$$f_1[Ga] = f_1[D(b_a(t_0))] \geqq f_1(b_a(t_0)) \geqq f_1(b_a(0)) = f_1(a),$$

while, if a is not an eigenelement, $b_a(t_0) \neq b_a(0) = a$,

$$f_1(b_a(t_0)) > f_1(a) \quad \text{and} \quad f_1[Ga] \geqq f(a).$$

Thus G is a deformation of P_c into the set of its eigenelements, not changing the eigenelements and increasing the values of f_1 on the remaining elements of P_c.

Suppose that $\lambda > c$.

LEMMA 1. *The critical set on the level surface $(f_1 = \lambda)$ in P_c is contained in the set A of eigenelements corresponding to the eigenvalue λ (all of these elements lie on $(f_1 = \lambda)$).*

In the course of the deformation G the set A does not change. For any $\epsilon > 0$ we may find an $\epsilon_1 > 0$ such that the elements of $S(A,\epsilon_1)$ remain after the operation G in $S(A,\epsilon)$.

Choose a $\delta > 0$ such that the eigenelements of $(\lambda - \delta \leqq f_1 \leqq \lambda + \delta)$ lie inside $S(A,\epsilon_1)$. Further, we may choose an $\alpha > 0$ such that the elements of P_c lying in the region $(\lambda - \delta \leqq f_1 \leqq \lambda + \delta)$ and outside $S(A,\epsilon_1)$ increase the values of f_1 under the operation G by

$$2\alpha = \min \{ f_1(Ga) - f_1(a) \},$$
$$a \in \{ P_c \times (\lambda - \delta \leqq f_1 \leqq \lambda + \delta) \},$$

(since the set on which the positive function $f_1(Ga) - f_1(a)$ is being considered is compact, so that its minimum 2α is positive). Suppose that α_1 is smaller than the two numbers α and δ. The entire portion of P_c lying in the region $(\lambda - \alpha_1 \leqq f_1)$ goes under the operation G into the portion of P_c lying in $(\lambda + \alpha < f_1) + S(A,\epsilon)$. In fact, if $a \in S(A,\epsilon_1)$, it remains in $S(A,\epsilon)$. If a lies outside $S(A,\epsilon)$, in $(\lambda - \alpha_1 \leqq f_1 \leqq \lambda + \alpha_1)$, then $f(a)$ receives an increment not less than $2\alpha_1$, and a goes into $(f_1 \geqq \alpha_1)$. If a was in $(f_1 \geqq \lambda + \alpha_1)$, then it remains in this region.

Thus the lemma is proved, because of the definition of critical set.

8. **Homology in P and its application.**

∇-*cycles*. Suppose given two elements $(a, -a)$ and $(b, -b)$ in P. The distance between them in P is the smaller of the numbers $\|a - b\|$, $\|a + b\|$. Suppose that this distance is less than $\epsilon < 1/2$, and, for example, $\|a - b\| < \epsilon$.

We denote by $f_i^\epsilon(a,b)$ for which $\|a - b\| < \epsilon$, equal to zero if the ith coordinates of the elements a and b have the same sign, and 1 if they have different signs. Since $f_1^\epsilon(a,b) = f_1^\epsilon(-a,-b)$, therefore f_1^ϵ is a function of pairs of elements of the projective space P, defining in it a one-dimensional ∇-cycle X_i^1. All the X_i^1 are ∇-homologous to one another.

Denote by (X^1) the family of all cycles ∇-homologous to X^1 in P. Here by ∇-homology in P we understand ∇-homology mod 2 in all P_c for all sufficiently small c.

The projective spaces z_n are n-dimensional Δ-cycles in all the P_c in which they lie.

Denote further by $X^n = X_1^1 \times X_2^1 \times \cdots \times X_n^1$ an n-dimensional ∇-cycle, the Alexander product of n cycles from the class (X^1). By (X^n) we denote the class of ∇-cycles ∇-homologous to X^n.

Suppose that the Δ-cycle x_n lies in P_c and $P_{c'}$. The intersection index $x_n \times X^n$ in P_c and $P_{c'}$ is the same. We shall therefore speak simply of the intersection index $x_n \times X^n$.

Denote by (x_n) the class of Δ-cycles of arbitrary P_c, for which $x_n \times X^n \neq 0$.

LEMMA 2. *The projective space* z_n *lies in the class* (x_n).

For $n = 1$ this is verified directly: $X_1^1 \times x_1 = 1$. We find

$$X^n \times z_n = (X_1^1 \times X_2^1 \times \cdots \times X_n^1) \times z_n.$$

Suppose that the ∇-cycle X_i^1 induces in the projective space z_n the 1-dimensional ∇-cycle $\overline{X}_i^1$, $i = 1, 2, \cdots, n$. Since $X_i^1 \times z_1 = 1$, the cycles $\overline{X}_i^1$ are 1-dimensional ∇-cycles in z_n, dual in it to the projective line. Therefore the 1-dimensional ∇-cycle $\overline{X}_i^1$ is equivalent in z_n to an $(n-1)$-dimensional projective hyperplane. And the intersection index $(\overline{X}_1^1 \times \overline{X}_2^1 \times \cdots \times \overline{X}_n^1) \times z_n$ in z_n is equal to the intersection index of n such $(n-1)$-dimensional hyperplanes in z_n, i.e., to 1.

Thus

$$X^n \times z_n = 1.$$

The lemma is proved.

LEMMA 3. *Each cycle* x_n *of the class* (x_n) *is homologous in* P *to the projective space* z_n.

Suppose that $x_n \in P_c$. For some c', $0 < c' < c$, $\bar{P}_{c'} \in z_k$, $k \geqq n$, and by a continuous deformation in $P_{c'}$, P_c goes into z_k. Here x_n goes into a Δ-cycle $\overline{x}_n$ of z_k, Δ-homologous to x_n in P_c. Since $X^n \times x_n = 1$, therefore $X^n \times \overline{x}_n = 1$, and this intersection index is also equal to 1 in z_k. Therefore x_n is nonhomologous to zero in z_k, but each n-dimensional Δ-cycle not homologous to zero in the projective space z_k is homologous to z_n in it. Thus $\overline{x}_n$ and hence x_n as well is homologous to z_n.

We turn to the proof of the fundamental theorem.

Let

$$\lambda_n = \sup_{x_n \in (x_n)} \min_{a \in x_n} f_1(a). \tag{15}$$

If $0 < c < \lambda_n$, then $(f_1 \geqq c)$ contains cycles x_n of class (x_n).

By a continuous deformation $(f_1 \geqq c)$ goes into a compact set P_c, while the values of the function $f_1(a)$ do not decrease under this deformation. The cycle x_n of $(f_1 \geqq c)$ goes into a cycle $\overline{x}_n$ of the same class (x_n), in P_c.

Denote by $(x_n)_c$ the collection of all cycles of (x_n) lying in P_c. The class $(x_n)_c$ is nonempty. Write

$$(\lambda_n)_c = \sup_{x_n \in (x_n)_c} \min_{a \in x_n} f_1(a). \tag{16}$$

We have

$$(\lambda_n)_c = \lambda_n.$$

In fact, since the class $(x_n)_c$ is a part of the class (x_n), it follows from (15) and (16) that

$$\lambda_n \geqq (\lambda_n)_c.$$

Further, in the definition of λ_n by formula (15) it is sufficient to restrict the x_n of (x_n) to those which lie in $(f_1 \geqq c)$. But under the deformation D each such x_n goes into an x_n' of the class $(x_n)_c$, with

$$\min_{a \in x_n'} f_1(a) \geqq \min_{a \in x_n} f_1(a)$$

since the values of the function f_1 do not decrease under the operation D. Hence $\lambda_n \leqq (\lambda_n)_c$.

Thus $\lambda_n = (\lambda_n)_c$ is a critical value of the function f on the compact set P_c, defined by the class $(x_n)_c$. Therefore $(f_1 = \lambda_n)$ must contain a nonempty critical set, which from Lemma 1 consists of eigenelements corresponding to the eigenvalue λ_n.

Thus the existence of the sequence λ_n is proved. Evidently

$$\lambda_{n+1} \leqq \lambda_n, \quad \lambda_n > c > 0.$$

We shall show that $\lambda_n \to 0$ as $n \to \infty$. Suppose on the contrary that there exists a $c > 0$ such that all $\lambda_n > c$. Then for any n the set $(f \geqq c)$, and thus P_c, would contain cycles of the class (x_n). But there exists a k such that P_c deforms into z_k under deformation into some $P_{c'}$ with $0 < c' < c$. Under this deformation the cycle x_{k+1} of the class (x_{k+1}), included in P_c, goes into a cycle $\bar{x}_{k+1}$ of the same class, lying in z_k. Hence for

$$X^{k+1} = (X_1^1 \times X_2^1 \times \cdots \times X_{k+1}^1), \quad X^{k+1} \times \bar{x}_{k+1} = 1$$

this product may be considered in the k-dimensional projective space z_k, and we arrive at the following contradiction: the product of $(k+1)$ one-dimensional ∇-cycles in z_k is necessarily homologous to zero, and that index must be equal to zero.

Thus the existence of such a number $c > 0$ leads to a contradiction and $\lambda_n \to 0$.

It remains to consider the case $\lambda_n = \lambda_{n+p} = \lambda$. Suppose that $0 < c < \lambda$. In P_c we have the following: λ is a critical value generated by the classes $(x_n)_c$ and $(x_{n+p})_c$. The class $(x_n)_c$ is subordinate to $(x_{n+p})_c$ by means of the ∇-class (X^p). Because of the general theorem, the critical set on

$(f = \lambda)$ contains a p-dimensional Δ-cycle x_p such that $x_p \times X^p = 1$. By Lemma 3, x_p is homologous to p-dimensional projective space z_p. We have proved the last part of Theorem 1 with the following sharpened statement:

If $\lambda_n = \lambda_{n+p} = \lambda$, *the eigenvalue* λ *corresponds to a set of normalized eigenelements, containing a p-dimensional cycle, homologous in P to p-dimensional projective space.*

9. **Odd operators.** Suppose that $f(a)$ now denotes an even and continuous, but generally speaking nonhomogeneous function, $f(-a) = f(a)$. Suppose that $f(a)$ is differentiable: $df(a;h) = (La, h)$. The operator La, generated by the differentiation of f, is a symmetric operator but odd: $L(-a) = -La$. We write $\psi(a) = (La, a)$. We take $\psi(a) > 0$ for $a \neq 0$, i.e., L is a positive operator, and in addition we suppose that $f(a) \to 0$ if $\psi(a) \to 0$ and that for each $\alpha > 0$ one can find a $\beta > 0$ such that if $\psi(a) < \alpha$ then $f(a) < \beta$. For $a = \theta$, $f(\theta) = 0$.

Application. If $\|a\| = 1$, $f(a) > 0$. In fact, if $0 < t \leqq 1$,

$$\frac{d}{dt} f(ta) = \frac{df(ta, dt \cdot a)}{dt} = df(ta, a) = \frac{1}{t}\, df(ta, ta) > 0.$$

$f(ta)$ increases for $0 < t \leqq 1$ and if $f(\theta) = 0$, $f(ta) \to 0$ as $t \to 0$, then $f(a) = f(1 \cdot a) > 0$.

As before, elements a for which $La = \lambda a$, $\|a\| = 1$, and λ is an eigenvalue will be called normalized eigenelements.

THEOREM 2. *Suppose that La is an odd symmetric positive completely continuous operator, satisfying the supplementary condition posed above. There exists a sequence of positive eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$, *where* $\lambda_n \to 0$ *and for the corresponding eigenelements* a_n *the numbers* $\mu_n = f(a_n)$ *form a nonincreasing positive sequence tending to zero. If* $\mu_n = \mu_{n+p} = \mu$, *there exists a set of eigenelements a for which* $f(a) = \mu$, *containing a p-dimensional* Δ*-cycle, homologous in P to p-dimensional projective space.*(¹)

The proof of this theorem repeats the proof of Theorem 1. Consider the region $(f > d > 0)$ in P. There exists a $c > 0$ such that this region is contained in $(\psi = (La, a) \geqq c)$. This last region may now be carried by a deformation analogous to the deformation D into some compact set

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(¹) V. Sobolev considered (in his dissertation [25]) a somewhat different class of odd operators.

P_c, containing all the eigenelements of the region $(\psi \geqq c)$, and thus also of $(f \leqq d)$, while the elements of P_c do not move under this deformation, and the values of $f(a)$ do not decrease. We write $P_d = P_c \times (f \geqq d)$. Thus, $(f \geqq d)$ goes into $\bar{P}_d$. Further, repeating all the preceding considerations with P_c replaced by the compact set P_d, we find that if

$$\mu_n = \sup_{x_n \in (x_n)} \max f(a),$$

then there are normalized eigenelements on $f = \mu$. If, for such an eigenelement, $La_n = \lambda_n a_n$, then $f(a_n) = \mu_n$ and $\psi(a_n) = L(a_n, a_n) = \lambda_n$, and $\lambda_n \to 0$.

Just as in the preceding case, $f(a_n) = \mu_n \to 0$, and $\lambda_n \to 0$ along with μ_n. Further if we have a merging $\mu_n = \mu_{n+p} = \mu$, the level surface $(f = \mu)$ contains a p-dimensional Δ-cycle not homologous to zero, consisting of eigenelements homologous in P to the projective space z_n.

EXAMPLE. Suppose that $K_i(s_1, s_2, \cdots, s_{2i})$ is a kernal for which

$$\int_0^1 \int_0^1 \cdots \int_0^1 K_i(s_1, s_2, \cdots, s_{2i}) \prod_{j=1}^{2i} \phi(s_j) \prod_{j=1}^{2i} ds_j > 0$$

for any ϕ for which

$$\int_0^1 \phi^2 ds = 1.$$

Consider the equation

$$\sum_{i=1}^{n} \int_0^1 \int_0^1 \cdots \int_0^1 K(s_1, s_2, \cdots, s_{2i-1}x) \prod_{j=1}^{2i-1} \phi(s_j) \prod_{j=1}^{2i-1} ds_j$$

$$= \lambda\phi(x)_2 \int_0^1 \phi^2 ds = 1.$$

There exists a countable set of "eigensolutions" $\phi_n(x)$ of this system and corresponding eigenvalues λ_n satisfying the above theorem.

CHAPTER III

Critical Sets of Geodesic Curves

10. Spaces R of curves with common endpoints on the sphere.

Fréchet metric. Suppose given in a metric space two arcs p and r which are images of $[0,1]$. We set up parametric representations of the curves p and r. To each point $t \in [0,1]$ there correspond points a_t and b_t on the curves p and r, continuously depending on t. The distance in the sense of Fréchet between the curves p and r is the lower limit with respect to all parametric representations of the maximum distances between pairs of points a_t and b_t of the curves p and r corresponding to common values of the parameter.

Suppose that m and n are closed curves, continuous images of the circumference. On m and n one may introduce cyclic coordinates ϕ with the same period 1, so that to each $\phi \in [0,1]$ there correspond points a_ϕ and b_ϕ of the curves m and n, with $a_0 = a_1$ and $b_0 = b_1$. The Fréchet distance $\rho(m,n)$ of these curves is the lower limit with respect to all possible parametric representations of the maxima of the distances between pairs of points a_ϕ and b_ϕ on the two curves with the same values of the cyclical coordinates.

In what follows R_n will denote an n-dimensional three-times differentiable closed manifold. The number $h = h(R_n)$ denotes a positive number depending on the manifold R_n such that a geodesic arc $\overline{ab}$ on R_n of length $\leqq h$ with endpoints a and b yields the minimum length for curves joining a and b.

Suppose that $\overset{\smile}{ab}$ is an arc joining the points a and b and of length $\leqq h$. The minimal geodesic arc $\overline{ab}$ joining a and b is called the chord spanning the arc $\overset{\smile}{ab}$. Suppose that $\alpha = \alpha(\delta)$ is a positive number depending on the manifold R_n and on the number $\delta > 0$, such that if the difference between the lengths of $\overset{\smile}{ab}$ and $\overline{ab}$ is less than δ, the Fréchet distance between $\overset{\smile}{ab}$ and $\overline{ab}$ is less than $\alpha = \alpha(\delta)$. If $\delta \to 0$, $\alpha(\delta) \to 0$. Thus to each manifold R_n there correspond the constant h and the function $\alpha(\delta)$.

Deformation of a family of curves. Suppose P is a family of segments or circumferences, given abstractly, such that one may establish a distance between curves $p \in P$ and points on them. To each point a_p of the curve p we relate a point $m(a_p, p)$ on R_n, where the function $m(a_p, p)$ continuously depends on a_p and p. The collection of points $m(a_p, p)$ for a fixed p and all possible a_p constitutes a curve $m(p)$, namely the image of p on R_n. The collection $m(P)$ of all curves $m(p)$ is called the image of the family P, defined by the function $m(a_p, p)$.

Now we assign to each point a_p of each curve p of P and to each value of the parameter $t \in [0,1]$ a point $m(a_p, p, t)$ on R_n, depending continuously on the point a_p, the curve p, and the parameter t. For a fixed t the functions $m(a_p, p, t)$ define on R_n a family of curves $m(P, t)$, the image of the family P. The transformation carrying $m(P, 0)$ into $m(P, 1)$ is called a continuous deformation of the family $P_0 = m(P, 0)$ as the image of the family P. If $m(a_p, p, 0) = a_p$, so that $P_0 = P$, we speak simply of deformation of the family P.

The space R of arcs on the sphere with common endpoints. Suppose that S_n is a Riemannian three-times differentiable manifold, homeomorphic to the n-dimensional sphere. By $R = R(S_n, a, b)$ we denote a metric space whose elements are rectifiable curves, images of straight lines, joining the points a and b on S_n, with the distance $\rho(p, q)$ between two curves p and q on R defined in the sense of Fréchet. Let $I(q)$ be a functional on R denoting the length of the curve q. The collection of all curves q in R for which the length does not exceed N, $I(q) \leqq N$, is a compact portion of R, denoted by R_N.

Suppose that p is a curve of R. We shall move along p in the direction from a to b. If s is the arc length described in moving from a along p to the point c of this curve, we say that this point c corresponds to a value of the parameter t equal to $s/I(p)$. Here the origin a of p corresponds to the value $t = 0$, and the endpoint b to the value $t = 1$. The parameter introduced in this way is called normal. The point of the curve P corresponding to the value t of the normal parameter is denoted by a_p^t.

A closed family B of curves of R is called a K-set if on the curves p of B the points a_p^t depend continuously on the curve p and the normal parameter t. For example, a closed family of geodesic polygons with a bounded number of sides or a closed family of curves with bounded curvature is a K-set.

Each K-set B may be considered as the image of some set $B_0 = m^{-1}(B)$ of segments $[0,1]$ of the real line, so that to the points 0 and 1 of these segments there correspond the points a and b. A segment whose image is a curve $p \subset B$ will be denoted by $m^{-1}(p)$. To the number t of the segment $m^{-1}(p)$ there corresponds the point a_p^t of the curve p. The topology on the segments is set up so that to a neighborhood of the curve p on B there corresponds a neighborhood of the corresponding segment $m^{-1}(p)$ on B_0. A deformation D of the family of curves B is a deformation of this family as an image of $m^{-1}(B)$.

If d is a double point of the curve p, i.e., it corresponds to two values of the normal parameter, t and t_1, $0 \leqq t \leqq t_1 \leqq 1$, then it goes generally speaking as a result of the deformation into two distinct points.

A K-deformation in R_N is a deformation of a K-set in R_N such that it goes anew into a K-set. A K-deformation in R is a K-deformation in some R_N.

11. **Critical sets in R.**

THEOREM 1. *The functional $I(q)$ in R has the K-property, and a K-critical set on $I = l$ is contained in the set A of geodesic arcs of length l joining the points a and b.*

In other words, for any $\epsilon > 0$ there exists an $\alpha > 0$ such that any K-set of $(I \leqq l + \alpha)$ may be deformed into

$$(I \leqq l - \alpha) + S(A, \epsilon).$$

Suppose given an arc $\overset{\smile}{cd}$ of length $< h$ and the chord $\overline{cd}$ spanning it.

We shall define a deformation carrying the arc $\overset{\smile}{cd}$ into $\overline{cd}$, as follows. We introduce on both arcs the normal parameter t so that the point c on both arcs corresponds to the value 0 and the point d to the value $t = 1$. We join points l_t and l_t' of the arcs corresponding to a common value of the parameter t by a minimal geodesic $\overline{l_t l_t'}$. The deformation which consists in moving the point l_t along the arc $l_t l_t'$ with a velocity proportional to the length of $\overline{l_t l_t'}$ into the position of l_t'. As a result $\overset{\smile}{cd}$ goes into $\overline{cd}$.

For each δ with $0 < \delta \leqq h$ we may find an $\alpha = \alpha(\delta)$ such that if $I(\overset{\smile}{cd}) - I(\overline{cd}) \leqq \delta$, then the entire path $\overline{l_t l_t'}$ is smaller in length than $\alpha(\delta)$, with $\alpha(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, the function $\alpha(\delta)$ depending on S_n.

If one is given a K-set M, and on the curves q of this set normal parameters, and if all of M lies in the region $(I < l + 1)$, then, taking $n > 4(l + 1)/h$, we obtain an arc $a_q^t a_q^{t+(1/n)}$ of the curve q corresponding to the interval $(t, t + (1/n))$ of the normal parameter, in length less than $h/4$. We shall take n to be even. On each curve $q \subset M$, we select $n + 1$ points

$$a_q^0 = a, a_q^{1/n}, \cdots, a_q^{i/n}, \cdots, a_q^{(n-1)/n}, a_q^1 = b.$$

We define an operation D as follows. Each arc

$$a_q^{j/n} a_q^{(j+2)/n}, \qquad j = 0, 2, 4, \cdots, n-2$$

of the curve will be deformed by the method described above into the chord spanning it.

After this deformation q goes into a geodesic polygon p. The points $a_q^{i/n}$, $i = 1, 3, \cdots, n-1$ go into the midpoints b_i of the sides of that polygon. Now we replace the arc $b_i b_{i+2}$ of the polygon p (consisting of two adjacent sides of the polygon) by the spanning chord $\overline{b_i b_{i+2}}$. We obtain a new polygon r. The passage from q to r defined in this way is the deformation D.

The deformation D does not change geodesic curves joining a and b.

Suppose that A is a set of geodesics joining a and b and of length l. For any $\epsilon > 0$ there exists an $\epsilon_1 > 0$ such that all curves lying in $S(A, \epsilon_1)$ remain after the deformation D in $S(A, \epsilon)$. (Curves sufficiently close to geodesic curves change arbitrarily little.) Further, for any $\epsilon_1 > 0$ there exists an $\epsilon_2 > 0$ such that all the geodesics joining a and b whose lengths lie within the limits $l - \epsilon_2$ and $l + \epsilon_2$ lie in $S(A, \epsilon_1)$.

If in the first part of the deformation D the curve $p (l - \epsilon_2 < I(p) < l + \epsilon_2)$ is changed by a sufficiently small number $\eta > 0$, it lies arbitrarily close to a geodesic polygon with n sides. In addition, if $\eta < l/2n$ then under the deformation D no arc corresponding to one of the intervals $(j/n, (j+2)/n)$ of the parameter t reduces to a point. If in fact such an arc did reduce to a point, the length I would have to go down by at least $l/2n$.

Suppose that after the first part of the deformation q goes into a polygon $p = Dq$.

If in the second part of the deformation D the length of the polygon decreased by a sufficiently small amount, this means that the angles of this polygon are arbitrarily close to π. Thus if the decrease of the length of the curve q under D is sufficiently small, the curve lies arbitrarily close to a geodesic polygon with sides which are not points, and whose angles are arbitrarily close to π. Such polygons in turn, with angles arbitrarily close to π, lie in an arbitrarily small neighborhood of a geodesic arc.

We may choose an $\alpha > 0$ which is arbitrarily small, and with $\alpha \leqq \epsilon_2$, so that every curve q of $(l - \epsilon_2 \leqq I \leqq l + \epsilon_2)$ which does not lie in $S(A, \epsilon_1)$

decreases in length under D by at least 2α. If q lies in $(l - \alpha \leqq I \leqq l + \alpha)$, then Dq falls in $(I \leqq l - \alpha)$. Curves of $S(A, \epsilon_1)$ remain after D in $S(A, \epsilon)$. Curves of $I \leqq l - \alpha$ remain in the same place after the operation D. Thus each K-set of $(I \leqq l + \alpha)$ goes into $(I \leqq l - \alpha + S(A, \epsilon))$. The theorem is proved.

12. **The space P of closed curves on the sphere.** Consider on S_n the collection P of all closed continuous images of the circle. We introduce a metric, defining the distance between curves q and q_1 of P in the sense of Fréchet. $I(q)$ is a functional in P denoting the length of $q \subset P$, P_N a compact portion of P defined by the inequality $I \leqq N$.

Each S_n may be considered as an n-dimensional Euclidean sphere with a given Riemannian metric. Suppose that B is the family of all the circumferences of that sphere.

We shall call a deformation of a portion of B normal if sufficiently close arcs of sufficiently close circumferences on the sphere go into arcs of curves arbitrarily close in length.

A K-set on P is the result of a normal deformation of a family of circumferences of B.

Suppose that A is a K-set, the result of a normal deformation D of a set of circumferences $B' \subset B$. We subject A to a deformation D_1 as the image of B', so that $D_1 D$ is again a normal deformation of B'. Such an operation D_1 is called a K-deformation. It carries a K-set into a K-set.

The deformation D'. We define the following normal operation. On the family of all circumferences B on S_n it is not possible to define the direction of traversing all the circumferences in a unique and continuous way. But for a sufficiently small $\epsilon > 0$ in the neighborhood $S(r, \epsilon)$ of a given circumference r it is possible in a continuous way to introduce a uniquely defined order of circuiting all circumferences entering into $S(r, \epsilon)$, and also to choose a point on each circumference of $S(r, \epsilon)$ in a continuous way.

A K-set A is a continuous image of some portion B' of the set B of circumferences: $A = D(B')$. Each curve $p = D(r)$ of A is the image of a circumference $r = D^{-1}(p) \subset B'$. To the neighborhood $U_\epsilon(r)$ of the circumference r: $U_\epsilon(r) = B' \times S(r, \epsilon)$ there corresponds a set $V_\epsilon(p) = D(U_\epsilon(r))$ of A. On all $p' = Dr'$ of $V_\epsilon(p)$, i.e., images of the circles $r' \subset U_\epsilon(r)$, one may uniquely and continuously establish a direction of circuit and choose a point on each, (continuous from the point of view of the topology of the preimage $U_\epsilon(r) = D^{-1} V_\epsilon(p)$). We shall now regard as the distance between the images $p = D(r')$ and $p_1 = D(r'')$ of two circles r' and r'' the distances between the preimages r' and r''. If one and the same curve p of A is the image of two circumferences r' and r'' of B', we will consider that curve as a pair of coincident images, with the distance between $p = D(r')$ and $p = D(r'')$ being regarded as distinct from zero. The continuity of the choice of points and directions on the curves p of A is understood relative to that metric, i.e., relative to the preimages $D^{-1}(p)$.

Suppose that on the family B' of circumferences there are chosen continuously both directions and points. Thus they are chosen continuously on the K-set $A'' = D(B'')$, in the sense mentioned above. Suppose that on p there have been chosen a point a_p and a direction of circuit. Choose on p a cyclic parameter ϕ such that the point a_p corresponds to the value $\phi = 0$ of the parameter. On moving along p from the point a_p in the direction of positive circuit of the curve p and describing an arc of length s, we attach to the endpoint of that arc the value $\phi = s/I(p)$ of the parameter. Such a parametrization is called normal. As ϕ goes from 0 to 1 we circuit the entire curve p.

The point a_p^ϕ denotes that point of the curve p corresponding to the given value of the normal cyclic parameter ϕ. We have $a_p^0 = a_p^1 = a_p$.

Suppose that n is an even number such that $I(p)/n < h/4$. We mark off on p a cyclic system of n points $a_p^0 = a_p, a_p^{1/n}, a_p^{2/n}, \cdots, a_p^{(n-1)/n}, a_p^1 = a_p$.

Now we carry out the following deformation of the curve p as the image of some circumference r. We contract each arc $a_p^{j/n}a_p^{(j+2)/n}$ with n even,

$$j = 0, 2, \cdots, n-2,$$

into the corresponding chord.

As a result the curve p goes into a geodesic polygon p' with $n/2$ sides and vertices $a_n^{j/n}$, with j even, $j = 0, 2, \cdots$. The points $a_p^{i/n}$, $i = 1, 3, \cdots, n-1$, odd, go into points b_i, the midpoints of the sides of the polygon p'.

Now we replace each arc $b_i b_{i+2}$, $i = 1, 3, 5, \cdots, n-1$, of the polygon p' by the chord spanning it. As a result of this operation p' goes into a new geodesic polygon p'' with vertices at the midpoints of the sides of p'. The successive application of two such operations to any curve q of the set $A' = D(B')$ is a K-deformation D' of the K-set A (we need only that $n > 4l/h$, where l is the maximum length of curves of B'). D' leaves unchanged the curve q if it is a closed geodesic, and otherwise always reduces the length. D' changes by arbitrarily little a curve q

which is sufficiently close to a closed geodesic, and, conversely, decreases by some number $\alpha = \alpha(\delta) > 0$ the lengths of curves which are distant from any closed geodesic by the distance $\delta > 0$.

13. **Critical sets in** P.

THEOREM 2. *The functional* $I(q)$ *in* P *has the K-property, and a K-critical set on* $(I = l)$ *in* P *consists of closed geodesics of length* l.

Now suppose that the K-set $A = D(B_1)$ is the image of a set of circumferences B_1, and suppose that l_1 is the maximum length of curves of A: $l_1 \geqq l > 0$. We denote by B_2 the portion of the set of circumferences B_1 consisting of elements of B_1 whose images on A have length $\geqq l/2$, i.e., $D(B_2) \subset (I \geqq l/2)$. For a sufficiently small $\eta > 0$ on each circumference of the neighborhood $U_\eta(r) \subset S(r, \eta)$ (2) of a circumference of B_2 one may choose a point and establish an order of circuiting, in a continuous way.

The closed set A_2 is covered by the system of neighborhoods $U_\eta(r)$. If m is the dimension of B, then one may choose the neighborhoods $U_\eta(r_i)$ such that no more than $(m + 1)$ of them have common elements. We choose a finite number of circumferences $r_1, r_2, \cdots, r_k$, such that the system $U_\eta(r_1), \cdots, U_\eta(r_k)$ covers A_2. Further, we choose larger neighborhoods $U_{\bar\eta}\,(r_i)$ such that in them, as before, we may choose a continuous system of circuiting the circumferences entering into them and of assigning a point on each circumference, and, as before, no more than $(m + 1)$ of them intersect.

We denote by V_η^i and $V_{\bar\eta}^i$ the images on A of the neighborhoods $U_\eta(r_i)$ and $U_{\bar\eta}(r_i)$.

On all the curves p of $V_{\bar\eta}^i$ we may choose a point a_p in a continuous way, and establish also an order of circuiting in a continuous way. Thus we may introduce in a continuous way a normal parameter ϕ. We denote by a_p^ϕ the point of the curve p corresponding to the parameter ϕ. The points $a_p^{j/n}$, $j = 0, 1, \cdots, n - 1$, form a cyclic system of points, chosen in a continuous way on the curves p of $V_{\bar\eta}^i$. For the curves p of the narrower neighborhood $V_{\bar\eta}^i$ we put

$$b_p^j = a_p^{(j+2)/n}.$$

Thus we have chosen continuously $n/2$ arcs $a_p^{j/n} b_p^j$ of equal length $2\,I(p)/n$ on all the curves p of V_η^i.

---

(2) $\bar{U}_{\bar\eta}$ and U_η are not necessarily spheres of radii $\bar\eta$ and η in B_2.

The choice of these arcs $a_p^{j/n}b_p^j$ may be extended to curves of a wider neighborhood $V_{\bar{\eta}}^i$ such that on the boundary of this neighborhood these arcs reduce to points $[b_p^j = a_p^{j/n}]$.

We shall denote by D_i operations which do not vary curves not entering into $V_{\bar{\eta}}^i$, and for curves p of $V_{\bar{\eta}}^i$ substituting first, for all the arcs $a_p^{j/n}b_p^j$ with j even, the chords spanning them, and then doing the same thing with the arcs $a_p^{i/n}b_p$ for odd i. For the curves of V_η^i this is the operation D' described above.

Now we apply successively to A the deformations $D_1, \cdots, D_k$.

Each curve p enters into no more than $(m+1)$ neighborhoods $V_{\bar{\eta}}^i$, so that it does not undergo more than $(m+1)$ deformations.

Curves which are closed geodesics of A do not change as a result of these operations. Curves sufficiently close to closed geodesics change by arbitrarily small amounts. For arbitrary $\epsilon > 0$ we can find an $\epsilon_1 > 0$, $\epsilon_1 = \epsilon_1(\epsilon)$, such that the curves which lie in $S(A^0, \epsilon_1)$, where A^0 is the set of all closed geodesics of S_n, remain in $S(A^0, \epsilon)$.

Further we can find an $\alpha > 0$ such that curves p of A which are of length greater than or equal to $l/2$ and lie outside $S(A^0, \epsilon_1)$ shorten their length under the operation $D_k D_{k-1} \cdots D_1$ by at least 3α.

Suppose that $l > 0$. Denote by A_l^0 the family of closed geodesics of length l. Fix $\epsilon > 0$ and write $\epsilon_1 = \epsilon_1(\epsilon)$. Further, choose an $\epsilon_2 > 0$ such that all closed geodesics of the region $(l - \epsilon_2 \leqq I \leqq l + \epsilon_2)$ lie in $S(A_l^0, \epsilon_1)$. Each curve lying in $S(A_l^0, \epsilon_1)$, as a result of the $(m+1)$ operations described above of shortening arcs into curves, remains in $S(A_l^0, \epsilon)$.

Further, we choose an α with $0 < \alpha < \epsilon_2$, such that curves of the region $l - \epsilon_2 \leqq I \leqq l + \epsilon_2$ and lying outside $S(A_l^0, \epsilon_1)$ decrease their length under the operation described above by at least 3α.

Suppose that B is any K-set included in $(I \leqq l + \alpha)$. For B we define the preceding operations, namely the choice of the neighborhoods V_η^i and the operations D_i, $i = 1, \cdots, k$. If $p \in B \times (l - \alpha \leqq I \leqq l + \alpha)$, then p enters into one of the V_η^i. As a result of the operations $D_1, D_2, \cdots, D_{i-1}$ the curve p either arrives in the neighborhood $S(A^0, \epsilon)$ but then remains under the following operations $D_i, \cdots, D_l$ in $S(A^0, \epsilon)$, or else p lies in $(l - \alpha \leqq I \leqq l + \alpha)$ but outside $S(A^0, \epsilon_1)$; in this case under the operation D_i the length $I(p)$ decreases by a number $\geqq 3\alpha$, and p falls into the region $(I \leqq l - \alpha)$ and remains there under all the following operations. Or else finally p falls into $(I < l - \alpha)$ and remains there under all the following deformations. Thus the deformation $D_k D_{k-1} \cdots D_1$ carries the arbitrary K-set B of $(I \leqq l + \alpha)$ into $(I \leqq l - \alpha) + S(A^0, \epsilon)$. Thus our theorem is proved.

REMARKS. 1. Introduce additional restrictions on K-deformations, as follows:

(1) A K-deformation does not change the curves of sufficiently small length $I \leqq \beta$ (the number $\beta > 0$ depends on the deformation).

(2) K-deformations do not convert curves q into isolated points (given that the curve q itself does not degenerate into a point).

(3) K-deformations do not convert any curve q into a twice (or more) covered simple arc (continuous image of an arc).

K-sets are the sets which are the results of K-deformations of closed sets of circumferences.

Under this narrower definition of K-sets Theorem 2 retains its validity.

We note that the operations $D_1, D_2, \cdots, D_k$ applied in the proof of Theorem 2 satisfy condition (1). But generally speaking they violate the new conditions (2)—(3) of the K-operations.

Closed geodesics and curves sufficiently close to them do not reduce to a point or to a k-times repeated image of a segment.

We consider now the neighborhood V_η^i. After the deformations $D_1, \cdots, D_{i-1}$ this neighborhood goes into some set (V_η^i). We write

$$B_i = V_\eta^i - S(A^0, \epsilon).$$

Suppose that $p \in B_i$. n arcs are chosen on p. The first part of the deformation D_i must map $n/2$ of these arcs into the chords spanning them. Suppose that a_p is the endpoint of one of these arcs. In preparation we carry out a new deformation of the curve p which consists of grafting onto the curve p and nearby a_p a loop with an angle at the point a_p. This loop may be chosen so that under the operation of shortening our $n/2$ arcs by chords the curve p with the little loop continues all the time and in the result of the deformation to satisfy conditions (2) and (3). The construction of these loops may be carried out in a neighborhood $W(p)$ of the curve p so small that on the boundaries of this neighborhood the little loops reduce to points. The grafting of these loops in the neighborhood $W(p)$ is a K-deformation. A finite number of such neighborhoods $W(p_j)$, $j = 1, \cdots, 2$, covers the set B_i, while these neighborhoods $W(p_1), W(p_2), \cdots, W(p_s)$ may be chosen so that no more than $(m+1)$ of them have common elements. In addition, we determine the length of a loop so as not to exceed for each curve the number $\delta/3(m+1)$, where δ was the number by which the length of the curve p decreased in the first part of the deformation D_i. Since on each curve not more than $(m+1)$ loops have been grafted, it follows that the lengthening of the curve p from the grafting of loops does not exceed $\delta/3$. In the first part of the operation D_i the curve p shortens by δ. As the result of the grafting of loops and the first part of the operation D_i the curve lengthens by a number $\geqq 2\delta/3$. Further, we carry out analogous graftings of loops onto the endpoints of one of the arcs with an odd index (contracted in the second portion of the operation D_i). If δ' is the shortening of the curve in the second portion of the operation D_i, then in the result of the grafting of the loops and the second part of this operation there occurs a shortening of p by a number $\geqq 2\delta'/3$. Earlier it was indicated that curves B lying outside $S(A^0,\epsilon)$ and thus outside $S(A^0,\epsilon_1)$ as well, shorten under the operation D_1 by a number $\geqq 3\alpha$, i.e., $\delta \geqq 3\alpha$. Now under the grafting of little loops there occurs after D_i a shortening of the curves p of B by a number $\geqq 2\delta/3 \geqq 2\alpha$, i.e., from the region $(I \geqq l + \alpha)$ they go as before into $(I \leqq l - \alpha)$, given of course that they were not already in the region $(I \leqq l - \alpha)$. All the preceding considerations remain valid if we replace the operation D_i by the operation D_i', differing from D_i by being preceded by the grafting of loops. In addition, the operations D_i' satisfy all the new conditions. The sequence of operations $D_1', D_2', \cdots, D_l'$ carry a K-set B of $(I \leqq l + \alpha)$ into $(I \leqq l - \alpha) + S(A^0,\epsilon)$. Thus Theorem 2 remains true if the K-sets and K-operations are understood in the new narrower sense.

REMARK 2. In the case $n = 2$ we may define K-sets as sets of nonselfintersecting curves obtained by K-deformations from a set of circumferences. Replacing the operation of contracting the arcs by the chords by a more complex operation defined in Chapter III of the monograph *Topological methods in the calculus of variations* by Ljusternik and Šnirel′man [19] (carrying a curve without double points into another). We may, without affecting any of the discussion or estimates of the preceding section, obtain the following theorem:

THEOREM 3. *A K-critical set on $P = P(S_2)$ on the level surface $I = l$ consists of closed non-selfintersecting geodesics of length l.*

We reproduce the proof of this assertion, as given in the monograph by Ljusternik and Šnirel′man.

Since the operation of reducing an arc to the chord spanning it generally converts a nonintersecting curve into a selfintersecting curve, we define the additional operation of "rectification", again carrying out a transformation of the curve into a non-selfintersecting one. For simplicity we begin with the case of a plane curve.

We shall consider the plane of the complex variable $z = x + iy$. Suppose that on it there is given a curve P, as in Figure 2, on which two points a and b with the complex coordinates α and β are given: $a(\alpha)$ and $b(\beta)$. These points decompose the curve P into two arcs $\overset{\smile}{ab}$ and $\underset{\frown}{ab}$. We suppose that the length of $\overset{\smile}{ab}$ does not exceed $d/4$, where d is the diameter of P.

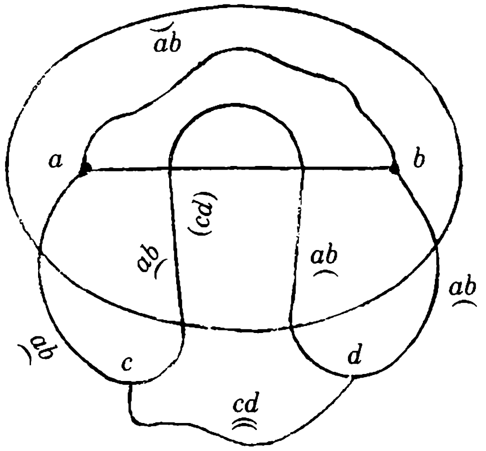

FIGURE 2.

We construct an ellipse E with foci at $a(\alpha)$ and $b(\beta)$ with a major axis equal in length to $\overset{\smile}{ab}$. The arc $\overset{\smile}{ab}$ lies inside E, while the portion $\underset{\frown}{ab}$ is situated outside E.

Consider the function

$$f(z) = u(x,y) + iv(x,y) = \ln\frac{z-\alpha}{z-\beta} + \pi i.$$

This function is multiple-valued, its imaginary part v being defined up to $2\pi k$, where k is an integer. We now define $f(z)$, and thus v, uniquely at all the points of $\underset{\frown}{ab}$, in the following way. At some point c of that arc, lying outside E, we suppose that $v(c)$ lies between 0 and 2π. Further, if d is an arbitrary point of $\underset{\frown}{ab}$, distinct from a and b, then, denoting by $\underset{\frown}{cd}$ the part of $\underset{\frown}{ab}$ lying between c and d, we put

$$f(d) = f(c) + \int_{\underset{\frown}{cd}} f'(z)\,dz = f(c) + \int_{\underset{\frown}{cd}} \left(\frac{1}{z-\alpha} + \frac{1}{z-\beta}\right) dz. \tag{1}$$

If the arc $\underset{\frown}{cd}$ does not intersect the chord $\overline{ab}$, then $v(d)$ also lies between 0 and 2π. In fact, $v(d)$ can equal zero only on the chord $\overline{ab}$.

LEMMA 1. *If the point d lies outside E, then $0 < v(d) < 2\pi$, even if the arc $\underset{\frown}{cd}$ intersects the chord $\overline{ab}$.*

In fact, the points c and d may be joined by an arc $\underset{\frown}{\underset{\frown}{cd}}$ lying entirely outside the ellipse E. The closed curve $\underset{\frown}{cd} - \underset{\frown}{\underset{\frown}{cd}}$ does not intersect $\overset{\smile}{ab}$. In fact, $\underset{\frown}{\underset{\frown}{cd}}$ lies outside and $\overset{\smile}{ab}$ inside the ellipse E. $\underset{\frown}{cd}$ and $\overset{\smile}{ab}$ have no common points since the curve P is non-selfintersecting. The points a and b lie therefore on one and the same piece of $\underset{\frown}{cd}$ into which the curve $\underset{\frown}{cd} - \underset{\frown}{\underset{\frown}{cd}}$ divides the plane. Therefore the integral

$$\int_{\underset{\frown}{cd} - \underset{\frown}{\underset{\frown}{cd}}} \left(\frac{1}{z - \alpha} - \frac{1}{z - \beta} \right) dz = 0,$$

or

$$\int_{\underset{\frown}{cd}} \left(\frac{1}{z - \alpha} - \frac{1}{z - \beta} \right) dz = \int_{\underset{\frown}{\underset{\frown}{cd}}} \left(\frac{1}{z - \alpha} - \frac{1}{z - \beta} \right) dz$$

and from (1)

$$f(d) = f(c) + \int_{\underset{\frown}{\underset{\frown}{cd}}} \left(\frac{1}{z - \alpha} - \frac{1}{z - \beta} \right) dz.$$

But since $\underset{\frown}{\underset{\frown}{cd}}$ does not intersect the chord $\overline{ab}$, it follows from the preceding discussion that $0 < v(d) < 2\pi$. The lemma is proved.

If ρ_1 and ρ_2 are the distances from some point $l(z)$ to $a(\alpha)$ and $b(\beta)$, then

$$u(l) = \ln|z - \alpha| / |z - \beta| = \ln \rho_1/\rho_2,$$

$$v(l) = \arg(z - \alpha) - \arg(z - \beta) + \pi i.$$

The curves $v = \text{const}$ are arcs of circumferences joining a and b. The curves $u = \text{const}$ are circumferences orthogonal to them. If the radii of the circumferences $u = \text{const}$ tend to zero, then their centers tend to a or b.

For a sufficiently small $\delta > 0$ the arcs $v = \delta$ and $v = 2\pi - \delta$ lie inside E on different sides of the chord $\overline{ab}$. It follows from Lemma 1 that for points d of the arc $\underset{\frown}{ab}$ lying outside E, $\delta < v(d) < 2\pi - \delta$.

Consider the function $\phi_\delta(\xi)$ of the real variable ξ: for

$$\delta \leqq \xi \leqq 2\pi - \delta, \quad \phi_\delta(\xi) = \xi;$$

for

$$\xi < \delta, \quad \phi_\delta(\xi) = \delta - (\delta/\pi) \arctan(\delta - \xi);$$

for

$$\xi > 2\pi - \delta, \quad \phi_\delta(\xi) = 2\pi - \delta + (\delta/\pi) \arctan(\xi - 2\pi + \delta).$$

$\phi_\delta(\xi)$ is a continuous and monotonically increasing function of ξ: for

$\xi \to \infty$, $\phi_\delta(\xi) \to 2\pi - \delta/2$; for $\xi \to -\infty$, $\phi_\delta(\xi) \to \delta/2$.

Now apply the operation D_1 of contracting the arc $\overset{\smile}{ab}$ into the chord $\overline{ab}$. The curve P goes into $D_1(P)$. We denote the differences of the lengths by

$$I(\overset{\smile}{ab}) - I(\overline{ab}) = \alpha \geqq 0.$$

Now we define a new deformation D_2^δ of the curve D_1P, namely an operation of rectification, as follows. The chord $\overline{ab}$ (into which the arc $\underset{\frown}{ab}$ had gone) remains invariant. We shall consider u and v as curvilinear coordinates of points on the plane. Each point $d(u,v)$ of the arc $\underset{\frown}{ab}$ distinct from a and b is carried into the point $(u, \phi_\delta(v))$.

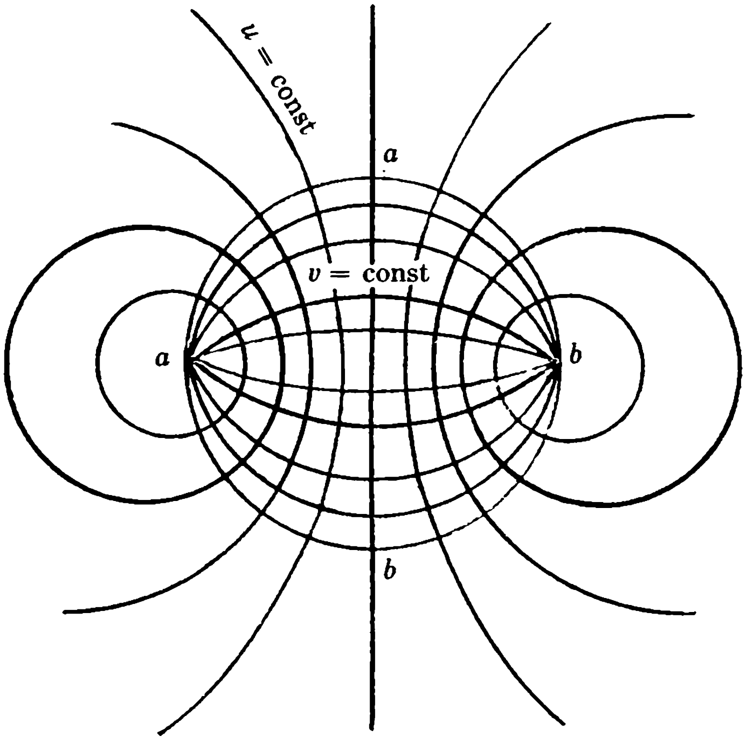

FIGURE 3.

In view of Lemma 1 and the definition of the function ϕ_δ all the points d go into points $D_2^\delta(d)$ lying outside $\overline{ab}$.

The transition from the point d to its image $D_2^\delta(d)$ is realized along the segment joining them.

The points d and $D_2^\delta(d)$ lie on one and the same circumference $u = \text{const}$. If the point d is sufficiently close to a or b, then the center of that circumference is arbitrarily close to a or b respectively, and the radius of the circumference is arbitrarily small. In a sufficiently small neighborhood of the point a or b the deformation D_2^δ displaces the point d

arbitrarily little. Since the points a and b themselves remain invariant under this operation, it follows that D_2^δ is continuous at these points and thus is a continuous deformation.

As a result of the deformation D_2^δ the arc $\underset{\frown}{ab}$ goes into the arc $D_2^\delta(\underset{\frown}{ab})$, which does not intersect the chord $\overline{ab}$.

As a result of the deformation $D = D_2^\delta D_1$ the curve P goes into the non-selfintersecting curve $P_1 = D(P)$.

After the deformation D_1 the length of the curve P decreases by the number

$$\alpha = I(\overset{\smile}{ab}) - I(\overline{ab}).$$

We shall show that for a sufficiently small δ the deformation D_2^δ increases the length of the curve by no more than $\alpha/2$.

Indeed, the linear element ds at the point z is equal to

$$ds = |dz| = \frac{|df(z)|}{|f'(z)|} = \frac{1}{|f'(z)|}\,|du + i dv|$$

$$= (du^2 + dv^2)^{1/2}\colon \left|\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right| = \frac{|z-\alpha|\,|z-\beta|\,(du^2+dv^2)^{1/2}}{|\beta-\alpha|}.$$

If the point z, for which $f(z) = u + iv$, goes into the point z_1 with $f(z_1) = u_1 + iv_1$, then the linear element ds goes into the linear element ds_1, where

$$ds_1 = \frac{|z_1-\alpha|\,|z_1-\beta|\,(du_1^2+dv_1^2)^{1/2}}{|\beta-\alpha|},$$

$$\frac{ds_1}{ds} = \left(\frac{du_1^2+dv_1^2}{du^2+dv^2}\right)^{1/2} \left|\frac{z_1-\alpha}{z-\alpha}\right| \cdot \left|\frac{z_1-\beta}{z-\beta}\right|.$$

Under the operation D_2^δ the point $(z) = (u, v)$ goes into $(z_1) = (u_1, v_1)$, where $u_1 = u$, $v_1 = \phi_\delta(v)$. Since $d\phi_\delta(v)/dv \leqq 1$, then

$$|du| = |du_1|, \quad |dv_1| \leqq |dv|$$

and

$$\frac{ds_1}{ds} \leqq \frac{|z_1-\alpha|\,|z_1-\beta|}{|z-\alpha|\,|z-\beta|}.$$

Since the points z and z_1 lie on the same circumference $u = \text{const}$, then

$$\left|\frac{z-\alpha}{z-\beta}\right| = \left|\frac{z_1-\alpha}{z_1-\beta}\right|$$

and

$$\frac{ds_1}{ds} \leqq \left| \frac{z_1 - \alpha}{z - \alpha} \right|^2 = \left| \frac{z_1 - \beta}{z - \beta} \right|^2 .$$

Suppose for example that $|z - \alpha| < |z - \beta|$. Consider the arcs $v = \delta$ and $v = 2\pi - \delta$ and the "lens" F bounded by them. If the point z lies on the circle $u = \text{const}$ outside F and is displaced under the deformation D_2^δ, then it goes into a point of the circle inside the lens, and its distance to the center α is decreased:

$$|z_1 - \alpha| < |z - \alpha| \quad \text{and} \quad ds_1 < ds.$$

If the point d lies on the circle $u = c$ inside the lens, then its distance from the point α may increase. The largest increase will occur if the point d lies on the chord $\overline{ab}$ and is displaced to the edge of the lens. But the relative increase will not exceed $1/\cos\delta < 1 + \delta^2$, i.e.,

$$ds_1/ds < 1 + \delta^2, \quad ds_1 < (1 + \delta^2)\, ds. \tag{2}$$

If $I(P)$ is the length of P, the length $D_1(P)$ will not exceed $I(P)$, and, from (2), the increase in the length of the curve under D_2^δ will not exceed $\delta^2 I(P)$.

If $\alpha > 0$, then, putting $\delta \leqq (\alpha/2I(P))^{1/2}$, we find that the increase in length of the curve under D_2^δ does not exceed $\alpha/2$.

If $\alpha = 0$, $\overset{\smile}{ab} = \overline{ab}$, the arc $\underset{\frown}{ab}$ does not intersect the chord $\overline{ab}$, and we may consider the rectification operation on the varying curve P.

The operation $D = D_2^\delta D_1$ has the following properties:

(1) It converts the curve P into a non-selfintersecting curve P_1.

(2) If $I(\overset{\smile}{ab}) - I(\overline{ab}) = \alpha$, then $I(P_1) - I(P) \geqq \alpha/2$.

(3) Points of the curve P sufficiently close to a or b are displaced by an arbitrarily small amount.

(4) The deformation leaves invariant the points of the curve outside the ellipse E. The portion of P lying inside E remains there throughout the course of the deformation.

(5) If the length of the arc $\overset{\smile}{ab}$ does not exceed the number $h > 0$, the displacement of each point of P in the course of the deformation D does not exceed $2h\alpha$.

Indeed, if the length of $\overline{ab}$ is equal to $2c$, then the length of $\overset{\smile}{ab}$ is equal to $2c + \alpha$, with $c \leqq c + (\alpha/2) \leqq h$.

The focal distance and the major and minor axes of the ellipse E are equal respectively to $c + (\alpha/2)$, c, and

$$\left(\left(c+\frac{\alpha}{2}\right)^2-c^2\right)^{1/2}=\left(\alpha c+\frac{\alpha^2}{4}\right)^{1/2}<(2h\alpha)^{1/2}.$$

The deformation D_1 displaces a point of the arc $\overset{\smile}{ab}$, dividing that arc in length in some ratio, into the corresponding point of the chord $\overline{ab}$, dividing the length of that chord in the same ratio. The greatest distance between corresponding points of the arc and the chord corresponds with the minor axis, and is achieved when the arc $\overset{\smile}{ab}$ consists of two equal segments $\overline{ad}$ and $\overline{db}$ and the point d corresponds to the midpoint of the chord $\overline{ab}$. Thus the displacement under the deformation D_1 does not exceed $(2h\alpha)^{1/2}/2$.

The displacement D_2^δ displaces a point lying inside E on the circumference $u=l$ to another point of the same circumference lying inside E. The displacement proceeds along a chord of the circumference $u=l$ lying in E, whose center lies on the prolongation of the major axis. The length of this chord does not exceed the length of the minor axis of E, i.e., $(2h\alpha)^{1/2}$.

(6) For a sufficiently small $\epsilon>0$ points p distant from a or b by no more than ϵ go into points distant from a or b by not more than 2ϵ.

D-deformation of a family of non-selfintersecting curves. Suppose that we are given a K-family (p) of non-selfintersecting plane curves p, whose diameters exceed $d>0$, and that on each curve p of this there has been selected a pair of points a_p, b_p with the complex coordinates α_p, β_p. Suppose further that these points divide each curve p into arcs $\overset{\smile}{a_pb_p}$ and $\underset{\frown}{a_pb_p}$, the length of a_pb_p being supposed not to exceed $d/4$. As above, we define for each curve p ellipses $E=E_p$, functions

$$f_p(z)=u_p+iv_p=\ln\frac{z-\alpha_p}{z-\beta_p}+\pi i,$$

numbers $\alpha=\alpha_p$, $\delta=\delta_p$, $\epsilon=\epsilon_p$, functions $\phi_{\delta p}(v_p)$, deformations D_1^p of reduction of the arc $\overset{\smile}{a_pb_p}$ into the chord $\overline{a_pb_p}$, deformations $D_2^\delta=D_2^{\delta p}$ unravelling the arc $\overset{\smile}{a_pb_p}$, and the complete deformation $D=D^p=D_2^{\delta p}D_1^p$.

The simultaneous application of the deformations D^p to the curves p of the family (p) is some K-deformation ϑ of this entire family. In order to prove this assertion, we need to establish the continuity of the deformation ϑ, i.e., that sufficiently close points of sufficiently close curves remain, under the deformation ϑ, arbitrarily close.

For the contracting deformation D_1^p this is evident. It remains to prove this continuity for the unravelling operation $D_2^{\delta p}$.

Suppose that $\bar{p}$ is some curve of (p). On the arc $\overset{\frown}{a_{\bar{p}}b_{\bar{p}}}$ we choose small arcs $\overset{\frown}{a_{\bar{p}}\alpha_{\bar{p}}}$ and $\overset{\frown}{\beta_{\bar{p}}b_{\bar{p}}}$ of length $\epsilon_{\bar{p}}/2$. The remaining portion $\overset{\frown}{\alpha_{\bar{p}}\beta_{\bar{p}}}$ of the arc $\overset{\frown}{a_{\bar{p}}b_{\bar{p}}}$ is distant from a and b by $\eta > 0$. Now we choose an arbitrarily small neighborhood $u_{\bar{p}}$ of the curve $\bar{p}$ on the family (p) such that:

(1) The numbers $\epsilon = \epsilon_p$ for the curves p of $u_{\bar{p}}$ are not less than $2\epsilon_{\bar{p}}$. Thus if on each arc $\overset{\frown}{a_p b_p}$ of the curve p of $u_{\bar{p}}$ we select small arcs $\overset{\frown}{\alpha_p a_p}$ and $\overset{\frown}{\beta_p b_p}$ of length $\gamma < \epsilon_p/2$, then the points of these small arcs are shifted under the operations $D_2^{\delta p}$ by distances not larger than 2γ (see property (6)).

(2) The remaining pieces $\overset{\frown}{\alpha_p\beta_p}$ of the arcs $\overset{\frown}{a_p b_p}$ are distant from the points a_p and b_p by $\eta_0 > 0$.

(3) The curve p of $u_{\bar{p}}$ is distant from $\bar{p}$ by a distance not exceeding ρ, where ρ is some number less than $\epsilon_p/2$.

If one considers the functions $f_p(z) = u_p + iv_p$, uniquely defining $v = v_p$ on the arcs $\overset{\frown}{a_p b_p}$ as above, for each curve p, then in view of property (3) the coordinates of v_p will, for all the arcs $\overset{\frown}{\alpha_p\beta_p}$ of the curves p of our neighborhood, be defined continuously in terms of the curve p and the point on the curve. Since the number δ_p depends continuously on the curve p, and the function $\phi_{\delta_p}(v)$ depends continuously on the curve, the deformation $D_2^{\delta p}$ defined by it will be a continuous deformation of the arc $\overset{\frown}{\alpha_p\beta_p}$ on our neighborhood u_p.

The remaining short arcs $\overset{\frown}{a_p\alpha_p}$ and $\overset{\frown}{\beta_p b_p}$, in view of conditions (1) and (2), remain during the deformation D_1^p in disks of radius $\leqq 2\gamma$ with centers a_p and b_p. Since the latter points were chosen continuously on the curves p, and the number γ may be taken arbitrarily small, we may then prove the continuity of the deformation D_2^p in the neighborhood of the curve $\bar{p}$. Thus the deformation D is a K-deformation. We have defined the "rectification" deformation for plane curves.

Suppose now given a K-family (p) of curves on the surface S_2. If this surface is smooth, then a sufficiently small piece of it, of diameter $\leqq h$, where h depends on S_2, may be approximated with arbitrary closeness by a plane. If on the curves p of this family continuous arcs $\overset{\frown}{a_p b_p}$ of length $\leqq h_0$ have been chosen, then we may construct around each such arc a geodesic ellipse E_p with foci at a_p and b_p and with the sum of the distances to the foci equal to the length of $\overset{\frown}{a_p b_p}$.

This ellipse will differ by arbitrarily little from the plane ellipse E_p. Inasmuch as the rectification deformation D_2^δ for plane curves only shifted those portions of them which lay inside the corresponding plane ellipse, it too may be defined for the part of the curve p which lies inside the geodesic ellipse E_p. For a sufficiently small h_0 all estimates in the form of inequalities introduced above for the plane case remain valid, and, defining the operation D as the successive application of the operations of contraction of arcs to chords and of the rectification for curves of the family (p) on the surface S_2, we preserve for this operation all the properties (1)—(4) formulated at the end of this section. Choosing such arcs $\overset{\smile}{a_p b_p}$ in a continuous way and applying the corresponding D-operations to the corresponding curves p, we obtain continuous deformations of the family (p). They are K-deformations.

Repeating now all the discussion applied in the proof of Theorem 2, with the contraction operations replaced by D-operations, we prove Theorem 3.

14. **Homotopy bases.** For compact families of non-selfintersecting curves on a two-dimensional sphere S_2 the following theorem holds.

Any compact family of non-selfintersecting curves on S_2 may be continuously deformed into a family of circumferences (Ljusternik [6], [7]).

The family A of all circumferences on the sphere S_2 is in this sense a homotopy basis for the collection of non-selfintersecting curves on S_2. This theorem was first proved for normal families in [6], and then for arbitrary compact families in [7].

Now we shall show that any compact family B of non-selfintersecting curves (in general nonrectifiable) may be transformed into a normal family, i.e., a family of rectifiable non-selfintersecting curves on which the length of the arc of a curve depends continuously on the arc and curve.

Suppose that B is a compact family of such curves. We shall prove that a sufficiently small neighborhood of any curve q of B may be transformed into a normal family.

Suppose that q lies in B, and that the point a lies inside one of the regions into which the sphere S_2 is divided by the curve q. Suppose that the distance from a to the curve q is equal to $\beta > 0$, with $0 < \alpha < \beta$. Then all the curves q' of the neighborhood U_q^α on the family B of curves q defined by the inequality $\rho(q, q') < \alpha$, have the point a in one of the regions into which they divide S_2, while these regions, which we denote by $O[q', a]$, vary continuously with q'. Considering S_2 as the complex plane with $O[q', \alpha]$ as a finite region of this plane, we may define a conformal mapping Tq' of the region $O[q', \alpha]$ into the unit disk, so that the point a goes into the center of this disk, and a fixed direction at a goes into a fixed direction. Thus the conformal mapping Tq' is defined uniquely. By a theorem of Courant, the conformal mapping Tq' depends continuously on q'. If we denote by $b(\phi, q')$ the point of the curve q' which under the mapping Tq' goes into the point $e^{i\phi}$ of the boundary of the unit disk, then the point $b(\phi, q')$ depends continuously on the cyclic coordinate ϕ and the curve q'. Thus in the neighborhood U_q^α of the curve q we may select in a continuous way cyclic parameters ϕ on all the curves with period 2π. Now we take a smaller neighborhood $U_q' = U_q^{\alpha/2}$. Denote by $[\phi_0, \phi_1; q']$ that arc of the curve q' which corresponds to the values of the interval $[\phi_0, \phi_1]$ of the cyclic parameter. Now we take the number n so large that the diameter of the arc

$$[\phi_0, \phi_0 + 2\pi/n, q']$$

is less than $h/4$. On the curves $q' \in U_{qk}^\alpha$, we select arcs

$$[0, \epsilon], \quad k = 0, 1, 2, \cdots, n-1,$$

where

$$\epsilon = 2\pi/n$$

if $q' \in U_q'$, i.e., if $\rho(q, q') < \alpha/2$,

$$\epsilon = (2\pi/n)\,(\alpha - \rho(q', q)),$$

if

$$\alpha \geqq \rho(q, q') \geqq \alpha/2.$$

On the boundary of U_q^α the arc just chosen reduces to a point. We obtain n arcs of diameter less than $h/4$ on each curve $q' \in U_q^\alpha$, which for each curve of U_q' fill out the entire curve. We apply a deformation carrying the arcs $[0, \epsilon; q']$ of the curves $q' \in U_q^\alpha$ into the chords spanning them, and then, if these arcs intersect the complementary pieces of q', we apply the unravelling operation mentioned above. This is a deformation D_0 of the family U_q^α. If we denote by D_k the analogous operation applied to the arcs

$$[2\pi k/n, (2\pi k/n) + \epsilon],$$

then the successive application of the operations $D_0, D_1, \cdots, D_n$ yields a deformation $D = D_q$ of the neighborhood which converts the smaller neighborhood U_q' into a normal family of rectifiable non-selfintersecting curves. On the boundary of U_q^α the curves do not change under the operation D_q. If we suppose that curves lying outside U_q^α do not change under D_q, then D_q is a continuous deformation of the entire family B.

Constructing such narrower neighborhoods U_q' and wider neighborhoods U_q^α, we choose a finite system of neighborhoods

$$U'_{q_1}, U'_{q_2}, \cdots, U'_{q_l},$$

covering the entire set B. The successive application to B of the deformations $D_{q_1}, D_{q_2}, \cdots, D_{q_l}$ is a deformation D^1, which converts B into a normal family B_1 of non-selfintersecting curves.

We shall call a family of non-selfintersecting curves obtained as a result of a normal deformation B_1 a K-set. Let $[B_1]$ be the class of such sets. If $c = \inf_{B_1' \in [B_1]} \sup_{q \in B_1'} I(q)$, then, from the considerations of §13, the level surface $(I = c)$ contains a nonempty set of non-selfintersecting closed geodesics of length c.

If S_2 is the unit sphere, these closed geodesics can be only great circles or isolated points, $c = 2\pi$ or 0. In view of the definition of c, for any $\alpha > 0$, in particular for $\alpha = \pi/2$, in the region

$$(I \leqq 2\pi + \alpha) = (I \leqq 2\pi/2)$$

there is a family B_2 of class $[B_1]$. B_2 is obtained as a result of some deformation D^2 of the set B_1.

We again construct on B_2 a neighborhood U_q^α of the curve $q \in B_2$, on which we may define in a continuous way the cyclic parameter ϕ, $0 \leqq \phi \leqq 2\pi$. We may in addition choose this parameter normal, in such a way that the lengths of the arcs $[0, \phi; q]$ of the curves q' of U_q^α are proportional to ϕ, i.e., so that their lengths are equal to $\phi / I(q')$. Choose on all curves $q' \in U_q^\alpha$ arcs $[0, \epsilon; q']$,

$$[2\pi/3, (2\pi/3) + \epsilon; q'], \quad [4\pi/3, (4\pi/3) + \epsilon; q'].$$

Indeed, for

$$q' \in U'_q,\ \rho(q', q) \leqq \alpha/2,\ \epsilon = 2\pi/3;$$

for

$$\alpha \geqq \rho(q', q) \geqq \alpha/2,$$

ϵ is defined according to the formula

$$\epsilon = (4\pi/3)\left[\alpha - \rho(q', q)\right].$$

ϵ depends continuously on $\rho(q', q)$ and $\epsilon = 0$ when $\rho(q', q) = \alpha$, so that on the boundary of U_q^α these arcs reduce to points. Each of these arcs does not exceed in length a third of the length of q (which does not exceed $5\pi/2$), i.e., the length of such an arc is strictly less than π.

Denote by E_q a deformation which does not alter curves lying outside U_q^α, and converts the three arcs chosen above on each curve q' of U_q^α

into the geodesic curves spanning them. Since the lengths of these arcs is less than π, we may carry out this operation E_q in a continuous way. We choose a finite system of curves $\overline{q_1}, \overline{q_2}, \cdots, \overline{q_r}$ of B_2 such that the neighborhoods $U'_{\overline{q_i}}$, $i = 1, 2, \cdots, r$ cover B_2. Denote by D^3 the deformation consisting in the successive application to B_2 of the operations

$$E_{\overline{q}_1}, E_{\overline{q}_2}, \cdots, E_{\overline{q}_{r1}}.$$

Suppose that the curve $q' \in B_2$ lies in some neighborhood $U'_{\overline{q}_i}$. The operation $E_{\overline{q}_i}$ converts this curve into a spherical triangle or into a great circle q''. The successive deformations $E_{\overline{q}_{i+1}}, \cdots, E_{\overline{q}_r}$ do not change q'', if q'' is a great circle, or, replacing portions of the perimeter of q'' by geodesic arcs if q'' is a spherical triangle, they change q'' into a convex spherical polygon lying in q'' (and thus in some hemisphere). Thus D^3 converts B_2 into the set B_3 of great circles and convex polygons.

Suppose that p is a polygon of B_3 lying on some hemisphere of S_2. Let d_p be the center of gravity of p (in the same hemisphere), g_p the largest circumference with center at d_p and lying in p (tangent to p), which we shall call "inscribed in p". If p approximates arbitrarily closely to the great circle p_0, the corresponding inscribed circumference also converges to p_0. We define a deformation D^4 of the set B_2 which does not alter the great circles of B_2 and carries every polygon p of B distinct from a great circle into the circumference g_p inscribed in p (to this end we need to shift each point b of p along the geodesic arc $\overline{bd_p}$ up to the intersection with g_p). B_3 is converted into a set B_4 of circumferences. The deformation $D^4D^3D^2D^1$ converts the original set of curves B into the set B_4 of circumferences.

CHAPTER IV

The Functional Space R of Curves with Common Endpoints on the Sphere

15. ∇-cycles in R.

One-dimensional ∇*-cycles.* Consider the space

$$R = R(S_2, a, b)$$

of rectifiable curves on the sphere S_2 with common endpoints at the points a and b and the curve-length functional $I(q)$ for curves q of S_2. We let R_N be the compact part of R defined by the inequality $I \leqq N$. We are taking the metric in R to be that of Fréchet; and without further discussion we are assuming that N is greater than the distance between a and b, so that R_N is nonempty. In all that follows we shall take as Δ-cycles in R those Δ-cycles which lie in R_N for a sufficiently large N. Δ-homology in R means homology in all the R_N for sufficiently large N. An n-dimensional ∇-cycle is homologous to zero in R if its products with all n-dimensional Δ-cycles are equal to zero. Thus ∇-homologies in R are defined.

Suppose that q_1 and q_2 are two elements of R_N with $\rho(q_1, q_2) < \epsilon$, where ϵ is a given small positive constant and $0 < \epsilon < h/2$. We may establish parametrizations on q_1 and q_2 such that points a_t and b_t of these curves corresponding to the same parameter t are distant from one another by a distance $< \epsilon$ and also

$$a_0 = b_0 = a, \quad a_1 = b_1 = b.$$

Suppose that

$$q_1, q_2 \in R_N, \quad \epsilon < \alpha/10N,$$

where α is the area of S_2. We join a_t and b_t by a minimal geodesic arc $\overline{a_t b_t}$. The family of all these arcs forms a film Q_1 with boundary $q_1 - q_2$. Suppose that c is a point of S_2 lying outside q_1 and q_2. We denote by $\Psi(1; q_1, q_2; c)$ the degree of covering of the point c by this film Q_1. The choice of parameter does not affect the value of this function provided that the corresponding points are at a distance from one another smaller than ϵ. We may replace the arcs $\overline{a_t b_t}$ by any continuous system of arcs

a_tb_t of length $\leqq 2\epsilon$. This will not change the index of the covering of the point c by the film formed by these arcs.

For any pair q_1, q_2 of arcs of R with $\rho(q_1, q_2) < \epsilon$ we write

$$f_1^\epsilon(q_1, q_2) = \Psi(1; q_1, q_2; c).$$

The sequence of functions f_1^ϵ as $\epsilon \to 0$ defines a one-dimensional ∇-complex, and since

$$f_1^\epsilon(q_1, q_2) + f_1^\epsilon(q_2, q_3) + f_1^\epsilon(q_3, q_1) = 0, \tag{1}$$

this complex is a one-dimensional ∇-cycle in R and any R_N. We shall denote it by T_c^1.

n-dimensional ∇-cycles. We turn to the film formed by the arcs $\overset{\smile}{a_tb_t}$ with endpoints on the curves q_1, q_2. The family of arcs $\overset{\smile}{a_{t'}b_{t'}}$ for $0 < t' \leqq t$ forms a film, and the degree of covering by this film of the point c is denoted by $\Psi_1(t; q_1, q_2, c)$. The arcs $\overset{\smile}{a_tb_t}$ are chosen so that only a finite number of these arcs pass through c. The function $\alpha(t) = \Psi_1(t; q_1, q_2, c)$ is an integer-valued function of t, having a finite number of jumps at values of t for which $\overset{\smile}{a_tb_t}$ passes through the point c. The corresponding jump $\alpha(t+0) - \alpha(t-0)$ is denoted by $d\alpha(t) = d\Psi_1(t; q_1, q_2; c)$. If $\overset{\smile}{a_tb_t}$ does not pass through c, this jump must be considered equal to zero.

Suppose we are given $(n+1)$ curves $q_1, q_2, \cdots, q_{n+1}$ of R, with $\rho(q_i, q_j) < \epsilon$ for all i and j. We establish parametrizations on these curves so that the points a_i^t of the curves q_i with common values of the parameter t are distant from one another by a distance $< \epsilon$. Suppose that $c_1, c_2, \cdots, c_n$ are n points of S_2 distinct from a and b and that the curves $q_1, \cdots, q_{n+1}$ do not pass through them. We write (recurrently)

$$\begin{aligned}&\Psi_n[t; q_1, q_2, \cdots, q_{n+1}; c_1, c_2, \cdots, c_n]\\ &\qquad = \int_0^t d\Psi_1(t'; q_1, q_2; c_1) \int_{t_1}^t d\Psi_{n-1}(t''; q_2, q_3, \cdots, q_{n+1}; c_2, c_3, \cdots, c_n).\end{aligned}$$

In expanded form for $t = 1$ we have

$$\begin{aligned}&\Psi_n(1; q_1, q_2, \cdots, q_{n+1}; c_1, c_2, \cdots, c_n)\\ &\qquad = \int_{t_1=0}^1 d\Psi_1(t_1; q_1, q_2; c_1) \int_{t_2=t_1}^1 d\Psi_1(t_2; q_2, q_3; c_2) \cdots\\ &\qquad\qquad \cdots \int_{t_n=t_{n-1}}^1 d\Psi_1(t_n; q_n, q_{n-1}; c_n).\end{aligned} \tag{2}$$

Since the differentials $d\Psi_1$ (the jumps) are different from zero only for a finite number of values of the parameters t_i (when the corresponding arcs pass through the points c_i), then the integral on the right side of (1) is equal to the sum of the products of the jumps

$$\sum_{0<t_1<t_2<\cdots<t_n<1} \prod_{i=1}^{n} d\Psi_1(t_i; q_i, q_{i+1}; c_i). \tag{3}$$

The sum is taken over those systems $t_1, \cdots, t_n$ for which all the jumps $d\Psi_1$ in the product are distinct from zero and $0 < t_1 < \cdots < t_n < 1$.

Now we choose on R a countable, everywhere dense net of elements ordered in some way. Suppose that $q_1, \cdots, q_{n+1}$ are elements of this net and that the order defined by the indices coincides with their relative order in the net. Suppose further that all the q_i lie in R_N and that they satisfy $\rho(q_i,q_j) < \epsilon$. We write

$$\begin{aligned} &f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}; c_1, c_2, \cdots, c_n) \\ &\qquad = f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}) = \Psi_n(1; q_1, q_2, \cdots, q_{n+1}; c_1, c_2, \cdots, c_n). \end{aligned} \tag{4}$$

If further $i_1, i_2, \cdots, i_n, i_{n+1}$ is a permutation of the indices $1, 2, \cdots, n, n+1$, and the function sign $(i_1, i_2, \cdots, i_n, i_{n+1})$ is taken to be $+$ or $-$ according as this permutation is even or odd, we may define

$$f_n^\epsilon(q_{i_1}, q_{i_2}, \cdots, q_{i_{n+1}}) = \operatorname{sign}(i_1, i_2, \cdots, i_{n+1})\, f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}). \tag{5}$$

REMARK 1. In what follows, in defining an n-dimensional ∇-complex by the functions $f_n^\epsilon(q_1, q_2, \cdots, q_n, q_{n+1})$, we shall give these functions for groups of elements $q_1, q_2, \cdots, q_n, q_{n+1}$ of the everywhere dense net chosen above, supposing that the q_i are ordered according to their order in that net. Each time we shall suppose without further discussion that on permutation of the elements $q_1, q_2, \cdots, q_{n+1}$ the function $f_n^\epsilon(q_1, q_2, \cdots, q_n)$ does not change or else only changes sign, depending on whether the permutation is even or odd.

REMARK 2. Suppose we have defined, using the preceding method, an n-dimensional ∇-complex Z^n by means of the functions $f_n^\epsilon(q_1, q_2, \cdots, q_n, q_{n+1})$, and suppose that these functions do not depend on one of the elements, for example on q_{n+1}. Then Z^n is an n-dimensional ∇-cycle which is homologous to zero. In fact Z^n is the ∇-boundary of the $(n-1)$-dimensional ∇-complex Z^{n-1} defined by the functions $\phi_\epsilon^{n-1}(q_1, q_2, \cdots q_n) = f_n^\epsilon(q_1, q_2, \cdots, q_n, q_{n+1})$.

LEMMA 1. *For a sufficiently small* ϵ *the function*

$$f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}; c_1, c_2, \cdots, c_n)$$

does not depend either on the choice of the parameter t on the curves q_1, $q_2, \cdots, q_{n+1}$ (provided that the points a_i^t of the curves q_i corresponding to a common value of the parameter t are distant from one another by less than ϵ), nor on the choice of the system of arcs $a_i^t a_{i+1}^t$, provided that these arcs do not exceed 2ϵ in length.

Suppose that 2α is a number smaller than the smallest distance between the points $a, b, c_1, c_2, \cdots, c_n$, and suppose that $2\epsilon < \alpha/10$. The disks $S(c_i, 2\epsilon)$, $i = 1, \cdots, n$ have no common points and do not contain the points a and b. Suppose that on the curves $q_1, q_2, \cdots, q_{n+1}$ parametrizations have been chosen so as to satisfy the conditions set forth above. If $d\Psi_1(t_i; q_i, q_{i+1}; c_i)$ differs from zero the curves q_i and q_{i+1} intersect the disk $S(c_i, 2\epsilon)$ and accordingly all the remaining curves $q_1, \cdots, q_{n+1}$ intersect $S(c_i, \alpha)$. This means that each of the curves $q_1, q_2, \cdots, q_{n+1}$ intersects all the disks $S(c_i, \alpha)$ if (3) is different from zero. We denote by $[\beta, \gamma; q_i]$ the part of the curve q_i which corresponds to the interval $[\beta, \gamma]$ of values of the parameter. We shall call the interval $[\beta, \gamma]$ an i-interval if the arc $[\beta, \gamma; q_i]$ lies in $S(c_i, \alpha)$, intersects $S(c_i, 2\epsilon)$ and the endpoints a_i^β, a_i^γ of this arc lie on the boundary of $S(c_i, \alpha)$. Since $[\beta, \gamma; q_i] \in S(c_i, \alpha)$, $\rho(q_i, q_j) < \epsilon$, and $\rho(c_i, c_j) > 2\alpha > \alpha + 2\epsilon$, then the arcs $[\beta, \gamma; q_i]$, $j \neq i$, lie outside $S(c_j, \alpha)$. Therefore the i-interval $[\beta, \gamma]$ and the j-interval $[\beta_1, \gamma_1]$, $i \neq j$, do not intersect. If the product of jumps from (3)

$$\prod_{i=1}^{n} d\Psi_1(t_i; q_i, q_{i+1}; c_i) \neq 0; \; 0 < t_1 < t_2 < \cdots < t_n < 1,$$

then t_i lies in some i-interval $[\beta_i, \gamma_i]$, since otherwise the arc $a_i^{t_i} a_{i+1}^{t_{i+1}}$ would not contain the point c_i. Since for $i < j$ the i-interval $[\beta_i, \gamma_i]$ and the j-interval containing $t_j > t_i$ do not intersect, then $\gamma_j > \beta_j > \gamma_i > \beta_i$ of 1-intervals $[\beta_1, \gamma_1]$, 2-intervals $[\beta_2, \gamma_2]$, and finally n-intervals $[\beta_n, \gamma_n]$, where $\beta_1 < \gamma_1 < \beta_2 < \cdots < \beta_n < \gamma_n$, are called admissible systems of i-intervals, $i = 1, 2, \cdots, n$.

We have: the right side of (2) is equal to

$$\sum{}' \int_{t_1=\beta_1}^{\gamma_1} d\Psi_1(t_1; q_1, q_2; c_1) \int_{t_2=\beta_2}^{\gamma_2} d\Psi_1(t_2; q_2, q_3; c_2) \cdots \cdots \int_{t_n=\beta_n}^{\gamma_n} d\Psi_1(t_n; q_n, q_{n+1}; c_n), \tag{2'}$$

where the sum $\sum'$ is extended over all admissible systems of i-intervals

$$[\alpha_1, \beta_1], \; [\alpha_2, \beta_2], \cdots, [\alpha_n, \beta_n].$$

Indeed, the product of jumps

$$\prod d\Psi_1(t_i; q_i, q_{i+1}; c_i), \quad 0 < t_1 < t_2 < \cdots < t_n < 1,$$

is different from zero only if all the t_i belong to an admissible system of i-intervals. Therefore the left side of (2) may be replaced by (2′), eliminating those systems of intervals of integration on which this product is zero.

But for an admissible system of intervals $[\beta_i, \gamma_i]$, $i = 1, \cdots, n$,

$$\prod_{i=1}^{n} \int_{t_i=\beta_i}^{\gamma_i} d\Psi_1(t_i; q_i, q_{i+1}; c_i)$$

does not depend on the choice of the arcs $\overset{\smile}{a_i^t a_{i+1}^t}$, provided that these arcs are less in length than 2ϵ. In fact, the film formed by those arcs when t runs through the i-interval $[\beta_i, \gamma_i]$ has the boundary

$$[\beta_i, \gamma_i; q_i] + \overset{\smile}{a_i^{\gamma_i} a_{i+1}^{\gamma_i}} - [\beta_i, \gamma_i; q_{i+1}] - \overset{\smile}{a_i^{\beta_i} a_{i+1}^{\beta_i}}.$$

The degree of covering of the point c_i by this film is equal to the integral

$$\int_{t_i=\beta_i}^{\gamma_i} d\Psi_1(t_i; q_i, q_{i+1}; c_i).$$

On the passage to a new system of arcs $\underset{\frown}{a_i^t a_{i+1}^t}$ of length less than 2ϵ our film is replaced by a new one whose boundary differs from the preceding one by having $\overset{\smile}{a_i^{\gamma_i} a_{i+1}^{\gamma_i}}$, $\overset{\smile}{a_i^{\beta_i} a_{i+1}^{\beta_i}}$ replaced by $\underset{\frown}{a_i^{\gamma_i} a_{i+1}^{\gamma_i}}$, $\underset{\frown}{a_i^{\beta_i} a_{i+1}^{\beta_i}}$. Since $\alpha = 10\epsilon$, the endpoints of this arc lie on the boundary of $S(c_i, \alpha)$, and their lengths are less than 2ϵ, this replacement will not change the degree of covering of the point c_i by the film, i.e., the value of our integral, which means also (2′) does not change the functions f_n^ϵ, which was to be proved.

Analogously one may prove the independence of the values of $\Psi_n^\epsilon[t; q_1, q_2, \cdots, q_{n+1}; c_1, \cdots, c_n]$, from the system of arcs, if t, $0 < t \leqq 1$, lies outside any i-interval.

We have proved that (2′) is independent of the choice of the system of arcs. The independence of (2′) of the choice of the parameter also reduces to this case. Suppose that on the curves $q_1, q_2, \cdots, q_{n+1}$ there have been chosen two systems of parameters t and τ, so that all the b_i^τ, $i = 1, 2, \cdots, n+1$ of curves q_i corresponding to a common value of the parameter τ (as well as the points a_i^t of the curves q_i corresponding to a common value of the parameter t), are distant from one another by a distance less than ϵ. Finally, suppose that we have drawn a system of geodesic arcs $\overline{b_i^\tau b_{i+1}^\tau}$ of length $\leqq \epsilon$ joining the points b_i^τ and b_{i+1}^τ.

Suppose on the curve q_i the transition from the parameter τ to t is defined by a monotone function $t = \phi_i(\tau)$. Then $b_i^\tau = a_{i+1}^{\phi_i(\tau)}$, $b_{i+1}^\tau = a_{i+1}^{\phi_{i+1}(\tau)}$. If $t = \phi_i(\tau)$, $t' = \phi_{i+1}(\tau)$, then, generally speaking $t \neq t'$, $\phi_{i+1}(\tau) \neq \phi_i(\tau)$. We denote by $a_i^t a_{i+1}^t$, $t = \phi_i(\tau)$ the arc

$$\overline{b_i^\tau b_{i+1}^\tau} + [\phi_{i+1}(\tau); \phi_i(\tau), q_{i+1}] = \overline{a_i^t a_{i+1}^t} + [t', t; q_{i+1}],$$

where the expression in square brackets denotes the portion of the arc q_{i+1} corresponding to the interval $[t, t']$ of values of the parameter t. The film formed by the arcs $a_i^t a_{i+1}^t$ coincides with the film formed by the arcs $\overline{b_i^\tau b_{i+1}^\tau}$. In fact, the arcs of the two systems differ only by portions of one and the same curve q_{i+1}, which is a boundary of the film. Hence we have reduced the effect of the change of parameter on a film to the effect of the change of the system of arcs under the same parametrization, i.e., to the case already proved. The lemma is completely proved.

The functions f_n^ϵ as $\epsilon \to 0$ define on R an n-dimensional ∇-complex $T^n_{c1,c2,\cdots,cn}$. We shall prove that it is a ∇-cycle.

We denote symbolically the multiple integral

$$\int_0^d d\Psi_1(t_1; q_{i_1}, q_{j_1}; c_1) \int_{t_1}^d d\Psi_1(t_2; q_{i_2}, q_{j_2}; c_2) \cdots \int_{t_{n-1}}^d d\Psi_1(t_n; q_{i_n}, q_{j_n}; c_n)$$

by

$$(q_{i_1}, q_{j_1})(q_{i_2}, q_{j_2}) \cdots (q_{i_n}, q_{j_n}). \tag{6}$$

The symbolic product (6) is the sum of ordinary products of the type (3) of jumps in the corresponding functions Ψ_1.

Lemma 2. *$T^n_{c1,c2,\cdots,cn}$ is an n-dimensional ∇-cycle.*

We need to prove for the $(n+2)$ elements $q_1, q_2, q_3, \cdots, q_n, q_{n+1}, q_{n+2}$ of R the equation

$$\sum_{i=0}^{n+1} (-1)^i (q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_i, q_{i+2})(q_{i+2}, q_{i+3}) \cdots (q_{n+1}, q_{n+2}) = 0,$$

(the first term on the left side $(i = 0)$ is $\prod_{j=2}^{n+1} (q_j, q_{j+1})$, and the last $(-1)^{n+1} \prod_{j=1}^{n} (q_j, q_{j+1})$). We shall prove this proposition by induction on n. For $n = 1$ it is true (see (1)):

$$(q_2, q_3) - (q_1, q_3) + (q_1, q_2) = 0. \tag{7}$$

Suppose that if $n = k$ the equation

$$\sum_{i=0}^{k} (-1)^i (q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_i, q_{i+2}) \cdots (q_k, q_{k+1}) = 0$$

holds for the $k+1$ elements $q_1, q_2, \cdots, q_{k+1} \in R$. Then, adding the superfluous factor (q_{k+1}, q_{k+2}), we have

$$(8) \qquad \sum_{i=0}^{k} (-1)^i (q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_i, q_{i+2})(q_k, q_{k+1})(q_{k+1}, q_{k+2}) = 0.$$

Subtracting the left side of (8) from

$$(9) \qquad \sum_{i=0}^{k+1} (-1)^i (q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_i, q_{i+2}) \cdots (q_k, q_{k+1})(q_{k+1}, q_{k+2}),$$

we obtain, since only the last two terms in (9) and the last term in (8) differ:

$$\begin{aligned} &\sum_{i=0}^{k} (-1)^i (q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_i, q_{i+2}) \cdots (q_k, q_{k+1})(q_{k+1}, q_{k+2}) \\ &\quad = (-1)^k (q_1, q_2)(q_2, q_3) \cdots (q_{k-1}, q_k)(q_k, q_{k+2}) \\ &\qquad + (-1)^{k+1} (q_1, q_2)(q_2, q_3) \cdots (q_{k-1}, q_k)(q_k, q_{k+1}) \\ &\qquad - (-1)^k (q_1, q_2)(q_1, q_3) \cdots (q_{k-1}, q_k)(q_{k+2}, q_{k+2}) \\ &\quad = (-1)^k (q_1, q_2)(q_2, q_3) \cdots (q_{k-1}, q_k) \{ (q_k, q_{k+2}) - (q_k, q_{k+1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (q_{k+1}, q_{k+2}) \} = 0 \end{aligned}$$

(in view of (7)). Thus our equation (6) is proved by induction from k to $k+1$, and the lemma is proved.

LEMMA 3. *If $c_1, c_2, \cdots, c_n$ and $d_1, d_2, \cdots, d_n$ are two systems of n points, different from a and b on S_2, then $T^n_{c_1c_2\cdots c_n} \overset{\sim}{\nabla} T^n_{d_1d_2\cdots d_n}$.*

We may deform the sphere S_2 into itself in a continuous and differentiable manner so that the points c_i, $i = 1, 2, \cdots, n$ go into the d_i, the space R goes into itself, and each R_N goes into a subset of some $R_{N'}$. Suppose that x_n is an n-dimensional Δ-cycle. It goes into a Δ-cycle x'_n, Δ-homologous to x_n. Suppose that N' is the maximum length of the images of x_n during the course of the deformation of x_n into x'_n. x_n is Δ-homologous to x'_n in each $R_{N''}$ for $N'' \geqq N'$. The intersection index $T^n_{c_1c_2\cdots c_n} \times x_n$ coincides with the intersection index $T^n_{d_1d_2\cdots d_n} \times x'_n$, but since $x'_n \overset{\sim}{\Delta} x_n$ the latter index is equal to $T^n_{d_1d_2\cdots d_n} \times x_n$. Thus, for any n-dimensional Δ-cycle x_n

$$(10) \qquad (T^n_{d_1d_2\cdots d_n} - T^n_{c_1c_2\cdots c_n}) \times x_n = 0.$$

Hence it follows that the difference on the left side of (10) is ∇-homologous to zero in any $R_{N'}$ for sufficiently large N', which proves the lemma.

We shall denote by T^n any of the mutually homologous ∇-cycles $T^n_{c_1 c_2 \cdots c_n}$.

For completeness of exposition we shall show that the classes of ∇-homologies defined above do not depend on the ordering of the everywhere dense net.

Functions g_n^i defining an n-dimensional ∇-cycle which is homologous to zero will be said to be equivalent to zero, and we shall write $g_n^i \approx 0$. The equation $h_n^i \approx g_n^i$ means that

$$h_n^i - g_n^i \approx 0.$$

We shall denote by $s_{ij} \, (i \neq j)$ the symbolic product

$$(q_1, q_2)(q_2, q_3) \cdots (q_{i-1}, q_i)(q_{i+1}, q_{i+2})(q_{i+2}, q_{i+3}) \cdots (q_j, q_{j+1}),$$
$$(q_j, q_{j+1})(q_{j+1}, q_{j+2}) \cdots (q_n, q_{n+1})$$

Here the factor $(q_i q_{i+1})$ is missing and (q_j, q_{j+1}) is twice repeated.

LEMMA 4. $s_{ij} \approx 0$.

For $i = 1$ or $i = n$ this follows from the fact that s_{1j} and s_{nj} do not depend on q_1 or q_n respectively. Suppose that $i < j$. Since

$$(q_{i-1}, q_i) = (q_{i-1}, q_{i+1}) - (q_{i-1}, q_i),$$

then

$$s_{ij} = -s_{i-1,j} + (q_1, q_2) \cdots (q_{i-2}, q_{i-1})(q_{i-1}, q_{i+1})(q_{i+1}, q_{i+2}) \cdots (q_n, q_{n+1}),$$

where the second term, not depending on q_i, is equivalent to zero. Hence $s_{ij} \approx s_{i-1,j}$, and further $s_{ij} \approx s_{i-1,j} \approx s_{i-2,j} \approx (-1)^{i-1} s_{1j} \approx 0$. Analogously, for $i > j$,

$$s_{ij} \approx s_{i-1,j} \approx s_{i+2,j} \approx (-1)^{n-i} s_{1j} \approx 0.$$

LEMMA 5.

$$(q_{i_1}, q_{i_2})(q_{i_2}, q_{i_3}) \cdots (q_{i_n}, q_{i_{n+1}}) \approx \operatorname{sign}(i_1, i_2, \cdots, i_n)(q_1, q_2)(q_2, q_3) \cdots (q_n, q_{n+1}). \tag{11}$$

It suffices to prove this lemma for the case of the transposition interchanging the pair of elements q_i and q_{i+1}. In this case (11) takes the form

$$\begin{aligned} &(q_1, q_2) \cdots (q_{i-2}, q_{i-1})(q_{i-1}, q_{i+1})(q_{i+1}, q_i)(q_i, q_{i+2})(q_{i+2}, q_{i+3}) \cdots (q_n, q_{n+1}) \\ &\qquad \approx -(q_1, q_2)(q_2, q_3) \cdots (q_{i-2}, q_{i-1})(q_{i-1}, q_i) \\ &\qquad\quad \cdot (q_i, q_{i+1})(q_{i+1}, q_{i+2}) \cdots (q_n, q_{n+1}). \end{aligned} \tag{12}$$

We have

$$(12') \qquad \begin{aligned} (q_i, q_{i+2}) &= (q_i, q_{i+1}) + (q_{i+1}, q_{i+2}) \\ (q_{i+1}, q_i) &= -(q_i, q_{i+1}) \\ (q_{i-1}, q_{i+1}) &= (q_{i-1}, q_i) + (q_i, q_{i+1}). \end{aligned}$$

Putting into the left side of (12), instead of the factors (q_{i-1}, q_{i+1}), (q_{i+1}, q_i), (q_i, q_{i+2}), their expressions from (12′), we replace it by several terms, of which one coincides with the right side and the remainder, from the preceding lemma, are equivalent to zero. Thus, for the case of transposition the lemma is proved. Successively applying the result for transpositions leads to a proof of the general case. It follows from Lemma 5 and the definition of the function f_n that the definition of the cycle $T^n_{c_1 c_2 \cdots c_n}$ does not depend, up to homology, on the ordering of the everywhere dense net.

16. **Δ-cycles in R.** Suppose that a and b are diametrically opposed points of the sphere S_2. We set up on S_2 a spherical system of coordinates, with a and b the poles and the semicircles joining these points as meridians, and one of these meridians as the zeroth. Longitudes will be expressed in radians. The longitude λ is a cyclic coordinate. We shall denote the meridian $\lambda = \lambda_0 =$ constant by (λ_0). As λ changes from 0 to 2π the meridian (λ) sweeps out the entire sphere. Taking the pole a as the origin and b as the endpoint, we establish on S_2 an orientation, coinciding with the orientation of the lune bounded by the pair of close meridians (λ_1) and (λ_2), $(\lambda_1) - (\lambda_2)$, where (λ_1) is circuited in the direction from a to b.

In what follows we shall suppose that the cyclic coordinate (λ) satisfies the condition $0 \leqq \lambda < 2\pi$, unless specified otherwise.

We shall now consider a curve q consisting of the meridians (λ_1), $(\lambda_2), \cdots, (\lambda_n)$ successively circuited beginning with the point a, so that one goes on (λ_1) from a to b and on (λ_2) from b to a. More generally on (λ_i) for i odd one goes from a to b and for i even from b to a, i.e.,

$$q = (\lambda_1) - (\lambda_2) + \cdots + (-1)^n(\lambda_n).$$

For n odd these curves have a as initial point and b as terminal point, i.e., they lie in $R(S_2, a, b)$.

Such a curve q is called an n-meridian or a polymeridian with n links, and denoted by $(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

We shall denote by t_n, for n odd, the collection of all n-meridians $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ for all possible values of the coordinates $\lambda_1, \lambda_2, \cdots, \lambda_n$.

For n even we mean by t_n the collection of all $(n+1)$ meridians $(\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_{n+1}^0)$ for all possible values of the coordinates $\lambda_1, \cdots, \lambda_n$ and a fixed value of $\lambda_{n+1} = \lambda_{n+1}^0$. The sets of curves thus defined for n odd and n even form subsets of $R = R(S_2, a, b)$ and constitute an n-dimensional Δ-cycle in R, and for a fixed n in all the R_N for sufficiently large N.

Each t_n is an n-dimensional torus, the $\lambda_1, \lambda_2, \cdots, \lambda_n$ being the cyclic coordinates on this torus.

Cycles in t_n. The torus t_n has n independent one-dimensional Δ-cycles $t_1^i\,(i = 1, 2, \cdots, n)$ given by the systems of $(n-1)$ equations $\lambda_j = \text{const.}$, where $j \neq i$.

The dual to t_1^i is the $(n-1)$-dimensional torus t_{n-1}^i, which is a Δ-cycle defined by the equation $\lambda_i = \text{const.}$ We have

$$t_{n-1}^i \times t_1^j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, equal to zero if $i \neq j$ and equal to unity if $i = j$.

Remark. In the case of even n, putting the fixed coordinate $\lambda_{n+1}^0 = 0$, we will for short write $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ instead of $(\lambda_1, \lambda_2, \cdots, \lambda_n, 0)$.

We shall relate to this $(n-1)$-dimensional Δ-cycle the one-dimensional ∇-cycles equivalent to it.

Consider two points in t_n: $q_1(\lambda_1, \cdots, \lambda_n)$ and $q_2(\bar{\lambda}_1, \cdots, \bar{\lambda}_n)$, the distance between which is less than some fixed ϵ (sufficiently small: $\epsilon < h$), so that

$$|\bar{\lambda}_i - \lambda_i| < \epsilon, \quad (i = 1, 2, \cdots, n).$$

We shall denote by $F_1^\epsilon(q_1, q_2)$ the function of the pair of points q_1, q_2 of the torus t_n for which $\rho(q_1, q_2) < \epsilon$,

$$q_1 = (\lambda_1, \lambda_2, \cdots, \lambda_n), \quad q_2 = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_n),$$

given by the following formulas. For simplicity we take $\lambda_i^0 \neq 0$. Then we put

$$F_i^\epsilon(q_1, q_2) = \begin{cases} 1, \text{ if } \lambda_i < \lambda_i^0 < \bar{\lambda}_i; \\ -1, \text{ if } \lambda_i > \lambda_i^0 > \bar{\lambda}_i; \\ 0, \text{ if } \lambda_i - \lambda_i^0 \text{ and } \bar{\lambda}_i - \lambda_i^0 \end{cases} \tag{13}$$

have the same sign.

The system of functions F_i^ϵ as $\epsilon \to 0$ defines a one-dimensional ∇-cycle, whose support is the $(n-1)$-dimensional torus t_1^{n-1}. We shall denote this cycle by Z_1^i. The cycle Z_1^i is dual to the one-dimensional Δ-cycle t_1^i, $Z_1^i \times t_1^j = \delta_{ij}$.

Therefore it follows that the ∇-cycle Z_1^i and the Δ-cycle t_{n-1}^i of complementary dimensions are equivalent.

Consider t_n as an n-dimensional cycle on itself. Then

$$(Z_1^1 \times Z_1^2 \times \cdots \times Z_1^n) \times t_n = 1$$

where the product inside the parenthesis denotes the Alexander product. This follows from the equivalence of Z_1^i and t_{n-1}^i and the equation $t_{n-1}^1 \times t_{n-1}^2 \times \cdots \times t_{n-1}^n = 1$ in t_n.

We shall give another expression for the functions F_i^i. On the polymeridians $q(\lambda_1, \cdots, \lambda_n)$ we define a parameter τ such that on transiting the meridian (λ_1) from a to b the value of τ changes from 0 to 1, on the transit of the meridian (λ_2) from b to a the value of τ changes from 1 to 2, and so forth. More generally on the corresponding circuit of (λ_i) τ changes from $i - 1$ to i. In total τ changes from 0 to n. For n even the supplementary interval $n \leqq \tau \leqq n + 1$ corresponds to the common fixed part of all the polymeridians of t_n. Under this assignment the points corresponding to a given value of τ on different polymeridians lie on the same parallel.

Suppose that $\rho(q_1, q_2) < \epsilon$. Join the points a_τ and b_τ of the curves q_1 and q_2 corresponding to the same value of the parameter τ by short arcs $a_\tau b_\tau$ of the parallel. We denote by $\Psi_1^i\ (\tau_1, \tau_2; q_1, q_2; c)$ the degree of covering of the point c by the film consisting of the arcs $a_\tau b_\tau$ for $\tau_1 \leqq \tau \leqq \tau_2$.

Lemma 6. *Suppose that the point a_i lies on the meridian (λ_i^0). We have*

$$\Psi_1^i(i - 1, i; q_1, q_2; a_i) = (-1)^{i-1} F_i^i(q_1, q_2)$$

On the curves $q_1(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $q_2(\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_n)$ the meridians $(-1)^i(\lambda_i)$ and $(-1)^i(\bar{\lambda}_i)$ correspond to the interval $(i - 1, i)$ of the parameter τ. Joining the points a_τ and b_τ for all $i - 1 \leqq \tau \leqq i$ by arcs $a_\tau b_\tau$, we obtain a film stretched on the lune formed by the meridians (λ_i) and $(\bar{\lambda}_i)$ with boundary $(-1)^{i-1}[(\lambda_i) - (\bar{\lambda}_i)]$. If $\lambda_i < \lambda_i^0 < \bar{\lambda}_i$ the point a_i lies on this film, and the orientation of its boundary coincides with the orientation of the sphere S_2, and the order of the point a_i relative to this film is equal to $(-1)^{i-1}$.

If $\lambda_i > \lambda_i^0 > \bar{\lambda}_i$, this film covers a_i, but the order of a_i relative to the film is $-(-1)^{i-1}$, since the orientation of $(-1)^{i-1}[(\lambda_i) - (\bar{\lambda}_i)]$ is opposite to the orientation of the sphere S_2.

In the remaining cases the points a_i lie outside the corresponding film, and the corresponding order is equal to zero.

Comparing the results just obtained with (13), we get

$$F_1^\epsilon(q_1, q_2) = (-1)^{i-1}\Psi_1^\epsilon(i-1, i; q_1, q_2; a_i).$$

The lemma is proved.

17. **Index of intersection of ∇- and Δ-cycles in** R. in the space R, for sufficiently large N, the ∇-cycles T^n and Δ-cycles t_n are defined in R_N. These n-dimensional cycles are dual. In fact:

THEOREM 1. $T^n \times t_n = (-1)^{n(n-1)/2}$.

To determine the intersection index $T^n \times t_n$ in R_N, where $R_N \supset t_n$, we need the intersection index in the manifold of the n-dimensional ∇-cycle $\overline{T}^n$ induced by T^n and t_n itself, considered as an n-dimensional Δ-cycle.

But, as we shall now prove, in the torus t_n the cycle $\overline{T}^n$ coincides with

$$\overline{T}^n = (-1)^{n(n-1)/2} Z_1^1 \times Z_1^2 \times \cdots \times Z_1^n, \tag{14}$$

i.e., up to sign with the Alexander product of the n ∇-cycles Z_1^i defined in the last section. Our theorem will follow from (13) and (14).

Take an everywhere dense net in R_N containing an everywhere dense subnet on t_n. For groups $q_1, q_2, \cdots, q_{n+1}$ of semimeridians of t_n belonging to this net and with indices increasing along with the ordinal number of the curve as an element of this set, we compare the functions

$$f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}),$$

defining the cycle $\overline{T}^n$, induced by T^n on t_n, with the functions

$$F_n^\epsilon(q_1, q_2, \cdots, q_{n+1})$$

defining $Z_1^1 \times Z_1^2 \times \cdots \times Z_1^n$. The parameter τ on the polymeridians is chosen as in the preceding section. Because of Lemma 7 and the definition of the Alexander product, we have

$$\begin{aligned} F_n^\epsilon(q_1, q_2, \cdots, q_{n+1}) &= \prod_{i=1}^n F_i^\epsilon(q_i, q_{i+1}) \\ &= \prod_{i=1}^n (-1)^{i-1}\Psi_1^\epsilon(i-1, i; q_i, q_{i+1}; a_i) \\ &= (-1)^{n(n-1)/2} \prod_{i=1}^n \int_{i-1}^{i} d\Psi_1^\epsilon(\tau_i; q_i, q_{i+1}; a_i). \end{aligned} \tag{15}$$

Further (see (4) and (2))

$$f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}) = \int_0^n d\Psi_1^\epsilon(\tau_i; q_1, q_2; a_1) \int_{\tau_1}^n d\Psi_1^\epsilon(\tau_2; q_2, q_3; a_2) \cdots \tag{16}$$
$$\cdots \int_{\tau_{n-1}}^{\tau} d\Psi^\epsilon(\tau_n, q_n, q_{n+1}; a_{n+1}).$$

For the proof of the theorem we need to prove that the right sides of (15) and (16) coincide.

Choose points a_i, $i = 1, 2, \cdots, n$, lying on different meridians (λ_i), and numbers $\alpha > 0$ and $\epsilon > 0$ such that no meridian intersects two of the disks $S(a_i, \alpha)$. We also take $\epsilon \leqq \alpha/4$. Consider $(n+1)$ polymeridians q_i, $i = 1, \cdots, n+1$, where $S(q_i, q_0') < \epsilon$. Under these hypotheses, it is easy to verify the coincidence of the integrals on the right sides of (15) and (16). In fact, consider for $\tau \in [i-1, i]$ the arcs $a^\tau a_{i+1}^\tau$ joining the points of the curves q_i and q_{i+1} with common values of the parameter τ. If

$$d\psi_1^\epsilon(\tau_i, q_i, q_{i+1}, \epsilon) \neq 0,$$

the arcs $a_i^\tau a_{i+1}^\tau$ contain the point a; the curve q_i intersects the disk $S(a_i, \epsilon)$, and the remaining curves $q_1, \cdots, q_n$ intersect $S(a_i, \alpha)$. The product

$$\prod_{i=1}^n d\psi_1(\tau_i, q_i, q_{i+1}, a_i), \quad 0 < \tau_1 < \tau_2 < \cdots < \tau_n < n, \tag{17}$$

can be different from zero if all the curves q_j, $j = 1, \cdots, n+1$ intersect all the disks $S(a_i, \alpha)$, successively in the order of the indices τ_i. But since each of the curves q_j constituting the meridian intersects only one of these disks, it is evident in this case that the disk $S(a_i, \alpha)$ intersects the ith constituent meridian of these curves.

For a value of the parameter τ_i for which the product (17) is different from zero, the arc $a_i^{\tau_i} a_{i+1}^{\tau_i}$ contains a_i, so that τ_i lies in the interval $[i-1, i]$ corresponding to the ith meridian. Therefore, if in the integrals we replace the limits on the right side of (16) by the numbers $i-1$ and i, we do not change the value of that integral. Thus we have proved that the right sides of (15), and hence therefore their left sides, coincide. In view of what was said before the theorem is proved.

It follows from Theorem 1 that, for any modulus $p > 0$, the Δ-cycles t_n and ∇-cycles T^n are not homologous to zero.

LEMMA 6. *In the space $R = R(S_2, a, b)$ there exist no n-dimensional Δ-cycles linearly independent from t_n.*

Suppose that on S_2 the metric of the Euclidean sphere has been established. Suppose that r_1 and r_2 are the smaller and larger arcs of the great circle r joining a and b, and suppose that the direction from a to b along the arc r_1 is positive. Denote by r_{2n+1} the arc $r_1 + nr$, and by r_{2n+1} the arc $r_2 - nr$ (they consist of arcs going around the circumference respectively r_1 and r_2 and n times).(1) The arc r_k contains $(k-1)$ points conjugate to a, while b is not conjugate to a, i.e., r_k is a nondegenerate geodesic arc in the sense of Morse [21] of the $(k-1)$st order. The number of linearly independent n-dimensional Δ-cycles in R is not greater than the number of geodesic arcs of the kth order joining a and b, as Morse proved for the case of homologies mod 2. But Morse's proof remains valid also for homologies modulo any prime (and with any field of coefficients, which we shall consider below). Therefore in R there are no two linearly independent Δ-cycles of arbitrary dimension (since there exists a unique geodesic arc of any dimension joining a and b.

Since t_n is nonhomologous to zero, then any n-dimensional cycle depends linearly on t_n, i.e.,

$$z_n \sim 0 \text{ or } z_n \sim qt_n, \qquad 0 < q < p. \tag{18}$$

Lemma 7. *If Z^n is an n-dimensional Δ-cycle such that $Z^n \times t_n = q$, then*

$$Z^n \overset{\sim}{\nabla} (-1)^{n(n-1)/2} qT^n. \tag{19}$$

In fact, if

$$U^{(n)} = Z^n - (-1)^{n(n-1)/2} qT^n,$$

then from the hypotheses of the lemma and Theorem 1

$$U^{(n)} \times t_n = 0.$$

But for any n-dimensional Δ-cycle z_n, from (18),

$$U^n \times z_n = 0.$$

Therefore $U^n \overset{\sim}{\nabla} 0$, which proves the lemma. Every n-dimensional ∇-cycle is homologous to zero or to qT^n for some $0 < q < p$.

We have found the structure of all the groups of Δ- and ∇-homologies for any prime field. Now suppose that the Δ-homologies are taken with the K-additive group of real numbers mod 1 as coefficient group, and correspondingly the ∇-homologies "modulo 0", i.e., with the whole numbers as the additive coefficient group.

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(1) *Translator's note.* There is obviously a garble somewhere in this sentence. If the author means to employ the same notation as Morse [21] (see top of page 245) he means that r_{2n} is $r_1 + nr$ and r_{2n+1} is $-r_2 - nr$, the positive direction on r_2 being that from a to b.

Theorem 1 remains valid if it is formulated as follows: for any α with $0 < \alpha < 1$

$$T^n \times \alpha t_n = \alpha \neq 0,$$

which proves that T^n and αt_n are nonhomologous to zero.

Analogously one proves that every n-dimensional Δ-cycle is homologous to 0 or at αt_n, $0 < \alpha < 1$, and that every n-dimensional ∇-cycle is either homologous to 0 or to qT^n, where q is any whole number.

18. **Products of ∇-cycles in R. The product $T^n \times T^m$.** We introduce the following concept. Suppose given two systems of elements, for example the numbers $(1, 2, \cdots, n)$ and $(n+1, n+2, \cdots, n+m)$.

We shall call a permutation of the $(n+m)$ elements of both systems *admissible* if it preserves the relative order of succession for the elements of the first system and for the elements of the second system. For example, from the elements $(1,2)$, $(3,4)$ we may set up 6 admissible permutations: $(1,2,3,4)$; $(1,3,2,4)$; $(1,3,4,2)$; $(3,1,2,4)$; $(3,1,4,2)$; $(3,4,1,2)$. However, the permutation $(2,1,3,4)$ is not admissible, because it destroys the order of succession of the elements $(1,2)$ of the elements of the first system.

Now form all possible admissible permutations of the $(n+m)$ elements of the two systems $(1, 2, \cdots, n)$, $(n+1, n+2, \cdots, n+m)$ of n and m elements. We shall denote by $C_{n,m}$ the difference between the number even and odd among these admissible permutations. For example, of the 6 admissible permutations of the elements $(1,2)$, $(3,4)$ presented above, 4 are even: $(1,2,3,4)$; $(1,3,4,2)$; $(3,1,2,4)$; $(3,4,1,2)$, and two odd: $(1,3,2,4)$; $(3,1,4,2)$. Thus $C_{2,2} = 4 - 2 = 2$. In other words

$$C_{n,m} = \sum{}' \operatorname{sign}(i_1, i_2, \cdots, i_{n+m}),$$

where the sum $\sum{}'$ is taken over all admissible permutations $(i_1, i_2, \cdots, i_{n+m})$ of the elements $(1, 2, \cdots, n)$, $(n+1, n+2, \cdots, n+m)$.

THEOREM 2.

$$T^n \times T^m \overset{\sim}{\nabla} C_{n,m} T^{n+m}. \tag{20}$$

Now suppose that the cycles $T^n_{c_1 c_2 \cdots c_n}$ and $T^m_{c_{n+1} \cdots c_{n+m}}$ are defined according to formulas (4) and (5) by the functions f_n^{ϵ} and f_m^{ϵ}.

Because of the definition of the Alexander product for cycles, for the

$n + m$ elements $q_1, q_2, \cdots, q_n, q_{n+1}, q_{n+2}, \cdots, q_{n+m+1}$,

$$\rho(q_i, q_j) < \epsilon, \; i,j = 1, 2, \cdots, n + m + 1,$$

taken from the net and ordered according to their order in the net, one may construct the functions

$$\begin{aligned} &F_{n+m}^{\epsilon}(q_1, q_2, \cdots, q_n, q_{n+1}, \cdots, q_{n+m+1}) \\ &\quad = f_n^{\epsilon}(q_1, q_2, \cdots, q_{n+1}; c_2, c_2, \cdots, c_n) \cdot f_m^{\epsilon}(q_{n+1}, q_{n+2}, \cdots, q_{n+m+1}; c_{n+1}, c_{n+m}). \end{aligned}$$

Because of the definition of f_n^{ϵ} and f_m^{ϵ}, passing from writing them in the form of Stieltjes integrals to the form (3) as the product of jumps

$$\begin{aligned} &F_{n+m}^{\epsilon}(q_1, q_2, \cdots, q_n, q_{n+1}, q_{n+2}, \cdots, q_{n+m+1}) \\ (21) \qquad &\quad = \sum{}' \prod_{i=1}^{n} d\Psi_1^{\epsilon}(t_i, q_i, q_{i+1}, c_i) \cdot \prod_{j=1}^{m} d\Psi_1^{\epsilon}(t_{n+j}, q_{n+j}, q_{n+j+1}, c_{n+j}), \\ &\quad 0 < t_1 < t_2 < \cdots < t_n < 1, \; 0 < t_{n+1} < t_{n+2} < \cdots < t_{n+m} < 1, \end{aligned}$$

where the sum $\sum'$ is extended over all products of jumps different from zero, for which the values of t_i, t_{n+j} satisfy the corresponding inequalities.

The right side of (19) may be represented in the form

$$(22) \qquad \sum_{(i_1, i_2, \cdots, i_{n+m})}{}'' \left\{ \prod_{k=1}^{n+m} d\Psi_1^{\epsilon}(\tau_k, q_{i_k}, q_{i_{k+1}}, c_{i_k}) \right\}, \qquad 0 < \tau_1 < \tau_2 < \cdots < \tau_{n+m} < 1,$$

where $(i_1, i_2, \cdots, i_{n+m})$ is an admissible permutation of $(1, 2, \cdots, n)$, $(n + 1, n + 2, \cdots, n + m)$.

The sum $\sum''$ is taken over all admissible permutations of these $(n + m)$ elements, and for every admissible permutation the sum $\sum'$ is taken over all products of jumps which are different from zero for the values of the parameters τ_k satisfying the corresponding inequalities.

Turning from the expression in the form of sums to the form of Stieltjes integrals, we obtain

$$\begin{aligned} &F_{n+m}^{\epsilon}(q_1, q_2, \cdots, q_{n+m+1}) \\ &\quad = \sum_{i_1, i_2, \cdots, i_{n+m}}{}'' \int_0^1 d\Psi_1^{\epsilon}(\tau_1, q_{i_1}, q_{i_2}, c_{i_1}) \int_{\tau_1}^1 d\Psi_1^{\epsilon}(\tau_2, q_{i_2}, q_{i_3}, c_{i_2}) \cdots \\ (23) \qquad &\qquad \cdots \int_{\tau_{n+m-1}}^1 d\Psi_1^{\epsilon}(\tau_{n+m}, q_{i_{n+m}}, q_{i_{n+m+1}}, \overline{c}_{i_{n+m}}), \end{aligned}$$

where the sum $\sum''$ is taken over all admissible permutations.

The functions depending on the permutation $(i_1, i_2, \cdots, i_{n+m})$ and standing under the summation sign in the first part of (21), define a certain $(n+m)$-dimensional ∇-complex, which we shall denote by $T^{n+m}_{i_1, i_2, \cdots, i_{n+m}}$. It follows from (21) that:

$$T^n \times T^m = \sum_{i_1, i_2, \cdots, i_{n+m}} T^{n+m}_{i_1, i_2, \cdots, i_{n+m}}.$$

Hence

$$(T^n \times T^m) \times (t_{n+m}) = \sum''_{i_1, i_2, \cdots, i_{n+m}} T^{n+m}_{i_1, i_2, \cdots, i_{n+m}} \times t_{n+m}. \tag{24}$$

Each product in the right side of (24) is equal, because of arguments repeating the proof of Theorem 1 (§17), a product in t_{n+m} of a group of $(n+m)$ one-dimensional ∇-cycles:

$$\begin{aligned} &(-1)^{(n+m)(n+m-1)/2} Z_1^{i_1} \times Z_1^{i_2} \times \cdots \times Z^{i_{n+m}} \\ &\qquad = (-1)^{(n+m)(n+m-1)/2} \operatorname{sign}(i_1, i_2, \cdots, i_{n+m}). \end{aligned}$$

Substituting in (22), we obtain

$$\begin{aligned} (T^n \times T^m) \times t_{n+m} &= (-1)^{(n+m)(n+m-1)/2} \sum''_{i_1, i_2, \cdots, i_{n+m}} \operatorname{sign}(i_1, i_2, \cdots, i_{n+m}) \\ &= (-1)^{(n+m)(n+m-1)/2} C_{n,m}. \end{aligned}$$

From the last equation and from (19) it follows that equation (20) is proved.

Examples. From the elements $(1, 2, \cdots, m)$ and $(m+1)$ we may form $m+1$ admissible permutations

$$(1, 2, \cdots, i, m+1,\ i+1, \cdots, m), \quad i = 0, 1, 2, \cdots, m.$$

Here $\operatorname{sign}(1, 2, \cdots, i, m+1,\ i+1, \cdots, m) = (-1)^{m-i}$. Therefore

$$C_{m,1} = \sum_{i=0}^{m} (-1)^{m-i} = \begin{cases} 1 & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$

Thus,

$$T^{2k} \times T^1 \overset{\sim}{\nabla} T^{2k+1},$$

$$T^{2k+1} \times T^1 \overset{\sim}{\nabla} 0.$$

Further,

$$C_{n,2} = [n + 2/2].$$

Indeed, all admissible permutations of $(1, 2, \cdots, n)(n+1, n+2)$ have the form

$$(1, 2, \cdots, i, n+1, i+1, \cdots, j, n+2, j+1, \cdots, n), \quad 0 \leqq i \leqq j \leqq n$$

$$\operatorname{sign}(1, 2, \cdots, i, n+1, i+1, \cdots, j, n+2, j+1, \cdots, n) = (-1)^{2n-i-j} = (-1)^{i+j}.$$

Hence

$$C_{n,2} = \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^{i+j} = \sum_{i=0}^{n} \frac{1-(-1)^{n-i+1}}{2}$$

$$= \frac{n+3/2-(-1)^{n+1}/2}{2} = \left[\frac{n+2}{2}\right].$$

Thus

$$T^n \times T^2 \overset{\infty}{\nabla} [(n+2)/2]\, T^{n+2}.$$

19. Application to geodesic arcs.

Classes of K-cycles in R. We have assumed that the metric on S_2 was the metric of the Euclidean sphere. We may put any Riemannian metric on S_2. Then on the curves $q \in R$ there is defined the functional $I(q)$, the length of the curve q. For sufficiently large N, R_N contains the family t_n. On R there are defined l-dimensional ∇-cycles T^l and a class (Z^l) of ∇-cycles homologous to qT^l, where q is any integer if the homology is mod 0 and $0 < |q| < p$ if the homology is taken with respect to a sufficiently large prime number p and so forth. For $Z^l \subset [Z^l]$, $Z^r \subset [Z^r]$, $Z^{l+r} \subset [Z^{l+r}]$ we have

$$Z^l \times Z^r \overset{\infty}{\nabla} C_{n,r} Z^{l+r}.$$

Since $C_{n,1} = 1$ for n even and 0 for n odd, $C_{n,2} = [(n+1)/2]$, $Z^l \times Z^2 \not\sim 0$, if $p > l$ (homologies mod p)

$$(Z^l \times Z^2) \in [Z^{l+2}], \tag{25}$$

and if l is even

$$(Z^l \times Z^l) \in [Z^{l+1}]. \tag{25'}$$

Denote by (z_k) a K-class of k-dimensional Δ-cycles for which $Z^k \times z_k \neq 0$ for $Z^k \subset [Z^k]$, $z_k \subset [z_k]$. The torus t_k is a representative of this class, so that the class is not empty.

In view of (25) the class (z_k) is subordinated to the class (z_{k+2}) by means of the class $[Z^k]$, and for k even from (25′), the class $[Z^k]$ is subordinated to the class $[Z^{k+1}]$ by means of the class $[Z^1]$.

M. Morse [21] discovered a countable set of geodesic arcs joining the points a and b of S_2. We shall sharpen this result.

We denote by c_k the lower bound of the maxima of I on the cycles z_k of the class (z_k). We have $c_k \leqq c_{k+1}$.

THEOREM 3. (1) *To each* c_k *there corresponds a geodesic arc of length* c_k *and order* k.

(2) *The numbers* c_k *grow unboundedly and remain included between the terms of two arithmetic progressions.*

(3) *Always* $c_k < c_{k+2}$.

(4) *For* k *even, if* $c_k = c_{k+1} = c$, *all the geodesic arcs issuing from the point* a *pass through the point* b, *while the arcs of these geodesic arcs that are bounded by the points* a *and* b *have equal lengths* c.

The first of these propositions follows immediately from Theorem 1 of Chapter I and Theorem 1 of Chapter III.

The level surface $(I = c_k)$ contains a nonempty critical set, contained in the set of geodesic arcs of length c_k joining a and b.

The fact that there exists on $(I = c_k)$ a geodesic arc of order k, i.e., containing k points conjugate to a, follows immediately from the Morse theory, which proved that a geodesic arc generated by a class of k-dimensional cycles can only be of order k. Morse [21] restricted himself to homologies mod 2, but his proof carries over verbatim to the case of arbitrary modulus.

For the proof of (2) we note the following. Suppose that H is the maximum of the distances between points of S_2, and e and f are points of S_2 for which $\rho(e,f) = H$. Choose points a_i, $i = 1, 2, \cdots, k$ such that for odd indices the a_i lie arbitrarily close to e and for even indices lie arbitrarily close to f, so as to satisfy $\sum_{i=1}^{k} \rho(a_i, a_{i+1}) = nH - \epsilon$, where ϵ is an arbitrary positive number.

A curve q passing through $a_1, a_2, \cdots, a_k$ has a length not less than $nH - \epsilon$. Since for any cycle $z_k \in [z_k]$ and ∇-cycle $T^k_{a_1 a_2 \cdots a_k}$ the intersection index differs from zero, it follows that z_k and the support of $T^k_{a_1 a_2 \cdots a_k}$ have a nonempty set-theoretical intersection; z_k contains a curve q entering into the support of $T^k_{a_1 a_2 \cdots a_k}$ which successively goes around the k points $a_1, a_2, \cdots, a_k$. The length of such a curve is not less than $kH - \epsilon$. Therefore

$$c_k = \inf_{z_k \in (z_k)} \sup_{q \in z_k} (q) \geqq kH - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, therefore $c_k \geqq kH$.

On the other hand, we may map the sphere S_2 analytically onto itself in such a way that a pair of diametrically opposite points a' and b' go into a and b, and the meridians joining a' and b' go into certain curves, namely distorted meridians. Suppose that d is the maximum of the lengths of these curves. The polymeridians with poles at a' and b', forming a family t'_k of curves consisting of k or $k+1$ meridians, and the lengths of such curves does not exceed $(k+1)\,d$. The family k'_k of polymeridians goes into the family t_k, on which $I(q) \leqq (k+1)\,d$. But $t'_k \in [z_k]$, so that

$$\sup_{q \in t_k} I(q) \leqq (k+1)\,d$$

$$c_k = \inf_{z_k \in (z_k)} \sup_{q \in z_k} I(q).$$

Thus

$$c_k \leqq (k+1)\,d, \qquad kH \leqq c_k \leqq (k+1)\,d.$$

The case of merging c_k. Now we turn to an analysis of the case when

$$c_k = c_{k+i}.$$

First of all we shall show that $c_k < c_{k+2}$. In fact, if $c_k = c_{k+2} = c$, then $(I = c)$ contains a critical set of dimension 2. The class (z_k) is subordinated to the class (z_{k+2}) by means of the class $[Z^2]$. Because of the fundamental theorem, for $c_k = c_{k+2} = c$ the "level surface" $I = c$ contains a critical set of dimension 2. This means that there exists a set of geodesic arcs joining a and b and of length c, having dimension 2. But since the set of all such arcs cannot have dimension 2, this proves item (3) of the theorem.

Item (4) follows from the subordination of the classes (z_k) to the class (z_{k+1}) for k even. In the case $c_k = c_{k+1} = c$, $k = 2l$, $(I = c)$ contains a one-dimensional Δ-cycle of geodesic arcs, not homologous to zero. In this case these geodesic arcs of length c joining a and b must issue from a in all dimensions.

REMARK. We have restricted ourselves to only those Δ-cycles which are K-cycles. But the introduction of arbitrary Δ-cycles in R (into all the R_N) does not give us new homology classes. Suppose that $0 < \epsilon < h$. Consider the ϵ-simplex $(q_1, q_2, \cdots, q_{n+1})$. We may introduce on the curves q_i, $i = 1, 2, \cdots, n+1$, a parameter τ, $0 \leqq \tau \leqq 1$, such that the points

a_i^τ of the curves q_i corresponding to the common value of the parameter τ are distant from one another by a distance $\leqq \epsilon$. Construct an ordinary geodesic simplex with $(n+1)$ vertices a_i^τ, $i = 1, 2, \cdots, n+1$, and consider the point $c = c(\tau; \lambda_1, \lambda_2, \cdots, \lambda_{n+1})$ having on this simplex barycentric coordinates λ_i, $i = 1, 2, \cdots, n+1$, $\lambda_i \geqq 0$, $\sum_i \lambda_i = 1$. For fixed $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ the collection of points $c(\tau; \lambda_1, \lambda_2, \cdots, \lambda_{n+1})$ with $0 \leqq \tau \leqq 1$ forms a curve $q(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$. The collection of curves $q(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$ for all possible λ_i with $\lambda_i \geqq 0$ and $\sum_i = 1$ forms a "normal" n-dimensional ϵ-simplex in R with vertices $q_1, q_2, \cdots, q_{n+1}$. This simplex is a K-set in R. Any ϵ-complex may be approximated by an ϵ-complex consisting of normal ϵ-simplexes with the same vertices. Such a complex is a K-set. In particular, any Δ-cycle may be ϵ-approximated by a K-cycle, and the introduction of arbitrary cycles in R_N (and in R) does not give anything new from the point of view of homology classes in comparison with K-cycles.

20. **Some homotopy questions in R.** By generalizing the concept of the "length" of a cycle introduced by S. Frolov and L. Èl′sgol′c for an n-dimensional manifold, one may define the concept of the length of a Δ-cycle in a compactum.

The Δ-cycle x has the length l, written long $x = l$, if there exist l but no more ∇-cycles $X_1, X_2, \cdots, X_l$ in M such that

$$(X_1 \times X_2 \times X_3 \cdots \times X_l) \times x \neq 0.$$

THEOREM 4. *The category of the Δ-cycle x in M is not less than its length plus unity:*

$$\operatorname{cat} x \geqq \operatorname{long} x + 1.$$

In fact, suppose that $\operatorname{long} x = l$, $\operatorname{cat} x \leqq l$. Then there exist l closed pieces $m_1, m_2, \cdots, m_l$ of the set x such that $x = \sum_{i=1}^{l} m_i$, $\operatorname{cat} m_i = 1$. Since $\operatorname{long} x = l$, there exists l cycles $X_1, X_2, \cdots, X_l$ such that $(X_1 \times X_2 \times \cdots \times X_l) \times x \neq 0$. This means that the set-theoretical intersection of the supports of those cycles and x is not empty.

Since $m_i (i = 1, 2, \cdots, l)$ reduce under deformation to a point, then every Δ-cycle in m_i is homologous to zero. In view of the theorem on merging of cycles, there exists a ∇-cycle $\bar{X}_i$ homologous to X_i and lying outside m_i. Also, since

$$\bar{X}_1 \times \bar{X}_2 \times \cdots \times \bar{X}_l \overset{\sim}{\nabla} X_1 \times X_2 \times \cdots \times X_l,$$

therefore

$$(\bar{X}_1 \times \bar{X}_2 \times \cdots \times \bar{X}_l) \times x \neq 0.$$

But on the other hand, each $\bar{X}_i$ lies outside m_i, and their set-theoretical intersection lies outside $\sum_i m_i = x$, i.e., the set-theoretical intersection of the supports of $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_l$ and x is empty. The resulting contradiction proves the theorem. The proof repeats the proof of the theorem for the n-dimensional case, with the Pontrjagin theorem on contraction of cycles replaced by its generalization by P. S. Aleksandrov [2a] to the case of a compactum.

Theorem 5. *The category of the set t_n in R satisfies the inequality*

$$[(n+3)/2] \leqq \operatorname{cat} t_n \leqq n+1.$$

The family t_n of polymeridians immediately decomposes into $n+1$ parts (no one of them covering the sphere S_2) which reduce to one polymeridian, i.e., they have category 1. Therefore the left side of the inequality follows.

The right side of the inequality will follow from the formula

$$Z^n \times Z^2 \overset{\infty}{\nabla} C_{n,2} Z^{n+2}, \qquad C_{n,2} = [(n+3)/2] \neq 0,$$
$$Z^{2k} \times Z^1 \overset{\infty}{\nabla} Z^{2k+1}, \qquad Z^{2k+1} \times Z^1 \overset{\infty}{\nabla} 0$$

and Theorem 1.

In fact, suppose that n is even, $n = 2l$. Then

$$\underbrace{Z^2 \times Z^2 \times Z^2 \times \cdots \times Z^2}_{l \text{ times}} \overset{\infty}{\nabla} C_l Z$$

$$C_l = C_{2,2} \times C_{4,2} \times \cdots \times C_{2l-2,2}.$$

If the homologies are taken with a modulus $p > C_l$, then

$$(\underbrace{Z^2 \times Z^2 \times Z^2 \times \cdots \times Z^2}_{l \text{ times}}) \times t_{2l} = C_l \neq 0.$$

Therefore $\operatorname{long} t_{2l} \geqq l$.

The length of t_{2l} cannot be greater than l, since if we had a product of $l+1$ or more ∇-cycles, the sum of whose dimensions was equal to $2l$, then among them there would be at least two one-dimensional cycles, and the product of two one-dimensional ∇-cycles (since $C_{1,1} = 0$) is always ∇-homologous to zero.

For $n = 2l + 1$

$$\underbrace{Z^2 \times Z^2 \times \cdots \times Z^2}_{l \text{ times}} \times Z^1 = C_l z_{2l} \times z_l = C_l Z^{2l+1}$$

$$(\underbrace{Z^2 \times Z^2 \times \cdots \times Z^2 \times Z^1}_{l \text{ times}}) \times t_{2l+1} = C_l \neq 0.$$

Therefore

$$\operatorname{long} t_{2l+1} \leqq l + 1.$$

But the length of t_{2l+1} cannot be greater than $l+1$, as follows from arguments analogous to those for the even case. Thus

$$\operatorname{long} t_n = [(n+1)/2].$$

For N sufficiently large, so that t_n lies in R_N, we therefore have

$$\operatorname{cat}_{R_N} t_n \geqq \operatorname{long} t_n + 1 = [(n+3)/2].$$

But $\operatorname{cat}_R t_n = \operatorname{cat}_{R_N} t_N$ for sufficiently large N. In fact, a deformation of the set of polymeridians into some rectifiable curve q of R may be approximated for sufficiently large N by a deformation into this curve in the space R_N.

Thus the left side of the inequality is also proved completely.

In the space R there exist sets of arbitrarily large category.

21. **Cycles with real coefficients.** In this section, without further discussion, we shall consider homologies with the additive group of real numbers as coefficients.

Suppose that on the sphere S_2 there is distributed some mass T, with density everywhere zero or positive, and suppose that the total mass is equal to $\alpha > 0$. By the T-measure of a region we shall mean the measure of that portion of the mass of T which lies in that region. For simplicity we shall first suppose that the density of this mass is positive inside some region t homeomorphic to the disk and equal to zero outside this region.

Suppose that q_1 and q_2 are two curves of R with $\rho(q_1, q_2) < \epsilon$. Suppose that on them we have chosen a parameter t such that the arcs $a_t b_t$ joining points q_1 and q_2 with common values of the parameter t may be chosen to be smaller in length than ϵ. Denote by $\phi_1[1, q_1, q_2, T]$ the T-measure of the film formed by the arcs $a_{t'} b_{t'}$ for $0 \leqq t' \leqq t$. $\phi_1[1, q_1, q_2, T]$

is the T-measure of the portion of S_2 bounded by $q_1 - q_2$. The functions $f_1^\epsilon(q_1, q_2) = \phi_1[1, q_1, q_2, T]$ define a one-dimensional ∇-cycle Z_T^1 relative to the reals. As in the preceding case one may prove that all the Z_T^2 are ∇-homologous to one another. Denote the class of all ∇-cycles ∇-homologous to αA_T^1, $\alpha \neq 0$, by (Z^1).

Now suppose we are given n masses $T_1, T_2, \cdots, T_n$, having densities different from zero inside regions $t_1, t_2, \cdots, t_n$ homeomorphic to the disk.

Further, if $\rho(q_i, q_j) < \epsilon$ $(i, j = 1, 2, \cdots, n+1)$, write

$$\begin{aligned} &\phi_n^\epsilon[q_1, q_2, \cdots, q_{n+1}, T_1, T_2, \cdots, T_n] \\ &\quad = \int_0^1 d\phi_1[t_1, q_1, q_2, T_1] \int_{t_1}^1 d\phi_1[t_2, q_1, q_2, T_2] \cdots \int_{n-1}^1 d\phi_1[t_n, q_1, q_2, T_n] \end{aligned} \tag{26}$$

(the differentials do not still denote jumps, since all the $\phi_i[t_i, q_i, q_{i+1}, T_i]$ are continuous functions of the parameters t_i).

For sufficiently small ϵ the values of the function ϕ_n^ϵ does not depend on the choice of the system of arcs forming the film joining the curves q_i and q_{i+1}. The proof is the same as in Lemma 1 of §15.

The functions of the elements of an everywhere dense net

$$f_n^\epsilon(q_1, q_2, \cdots, q_n) = \psi_n^\epsilon(q_1, q_2, \cdots, q_{n-1}, T_1, T_2, \cdots, T_n), \tag{27}$$

if $q_1, q_2, \cdots, q_{n+1}$ are ordered as in the net, and

$$f_n^\epsilon(q_{i_1}, q_{i_2}, \cdots, q_{i_{n+1}}) = \operatorname{sign}(i_1, i_2, \cdots, i_{n+1}) f_n^\epsilon(q_1, q_2, \cdots, q_{n+1})$$

define an n-dimensional ∇-complex $Z_{T_1 T_2 \cdots T_n}^n$.

Analogously to the case of a prime modulus p one proves that:

(1) all the $Z_{T_1 T_2 \cdots T_n}^n$ are n-dimensional ∇-cycles;

(2) all the $Z_{T_1 T_2 \cdots T_n}^n$ are homologous to one another. The class of all cycles ∇-homologous to $Z_{T_1 T_2 \cdots T_n}^n$ is denoted by (Z^n).

(3) if $Z^n . Z^m, Z^{n+m}$ are cycles ∇-homologous to

$$Z_{T_1 T_2 \cdots T_n}^n, \ Z_{T_1 T_2 \cdots T_m}^m, \ Z_{T_1 T_2 \cdots T_{n+m}}^{n+m},$$

then

$$Z^n \times Z^m \overset{\backsim}{\nabla} C_{m,n} Z^{m+n}.$$

The hypertori t_n considered in §17 are also n-dimensional Δ-cycles when taken with real coefficients.

THEOREM 6. *The intersection index* $Z^n \times t_n = \alpha^n$.

In the case $n = 1$ this is proved directly. The set t_1 of meridians covers the sphere S_2.

If

$$q_0, q_1, q_2, \cdots, q_n = q_0, \quad \rho(q_i, q_{i+1}) < \epsilon$$

is a cyclic sequence of meridians, then

$$f_1^\epsilon(q_i, q_{i+1}) = \phi_1(q_i, q_{i+1})$$

is the T-measure of the strip bounded by $(q_i - q_{i+1})$ and

$$\sum_{i=0}^{r-1} f_1^\epsilon(q_i, q_{i+1})$$

is the T-measure of the entire sphere S_2. Therefore this measure is equal to α and

$$Z_T^1 \times t_1 = \alpha. \tag{28}$$

We turn to the general case. Here t_n is an n-dimensional torus, and its element the polymeridian $q(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is defined by the n cyclic coordinates $\lambda_1, \lambda_2, \cdots, \lambda_n$. We introduce a new parameter τ on the polymeridian $q(\lambda_1, \lambda_2, \cdots, \lambda_n)$ so that the ith meridian of the polymeridian q corresponds to the interval $(i-1, i)$ of that parameter; τ varies from 0 to n. Therefore in the definition of $f_n^\epsilon(q_1, q_2, \cdots, q_{n+1})$ by the formula (27), for the polymeridians q_i we need to put n as the upper limit of the integrals. Write

$$\begin{aligned} &\begin{bmatrix} \beta_1 & \beta_2 & & \beta_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \\ &\quad = \int_{\alpha_1}^{\beta_1} d\phi_1[t_1, q_1, q_2, T_1] \int_{\alpha_2}^{\beta_2} d\phi_1[t_2, q_2, q_3, T_2] \cdots \int_{\alpha_n}^{\beta_n} d\phi_1[t_n, q_n, q_{n+1}, T_n]. \end{aligned} \tag{29}$$

In particular

$$f_1^\epsilon(q_1, q_2, \cdots, q_{n+1}) = \begin{bmatrix} n & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix}.$$

We have

$$\begin{bmatrix} n & n & & n' \\ 0 & t_1 & \cdots & t_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & n & & n \\ 0 & t_1 & \cdots & t_{n-1} \end{bmatrix} + \begin{bmatrix} n & n & & n \\ 1 & t_1 & \cdots & t_{n-1} \end{bmatrix}.$$

But

$$\begin{bmatrix} n & n & n \\ 1 & t_1 & t_{n-1} \end{bmatrix} \approx 0. \tag{30}$$

In fact, the lower limits in the integrals (29) corresponding to the left side of (30) are all $\geqq 1$, i.e., if this integral is a function $g(q_1, q_2, \cdots, q_{n+1})$ of the vertices $q_1, q_2, \cdots, q_{n+1}$ of an n-dimensional simplex, where

$$q_i = (\lambda_1^i, \lambda_2^i, \cdots, \lambda_n^i),$$

then that function does not depend on the first coordinates λ_1^i of the vertices

$$q^i (i = 1, 2, \cdots, n+1),$$

since the first meridians λ_1^i of the polymeridians q_i correspond to the intervals $(0, 1)$ of the parameters t_i, which do not enter into the intervals of integration. If

$$q_i' (0, \lambda_2^i, \cdots, \lambda_n^i)$$

is the projection of q_i onto the $(n-1)$-dimensional manifold $\lambda_1 = 0$, then

$$g(q_1, q_2, \cdots, q_{n+1}) = g(q_1', q_2', \cdots, q_{n+1}').$$

Therefore the intersection index of the n-dimensional ∇-cycle generated by the function g with t_n coincides with its intersection index with the n-dimensional Δ-cycle t_n' constituted by the projections of the simplexes t_n on the manifold $\lambda_1 = 0$. This index is equal to zero, which proves (30). Therefore

$$\begin{bmatrix} n & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix} \approxeq \begin{bmatrix} 1 & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix}.$$

Analogously one proves

$$\begin{bmatrix} 1 & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix} \approxeq \begin{bmatrix} 1 & 2 & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix}$$

and more generally

$$\begin{bmatrix} 1 & 2 & \cdots & i-1 & n & n & & n \\ 0 & t_1 & \cdots & t_{i-2} & t_{i-1} & t_i & \cdots & t_{n-1} \end{bmatrix} \approxeq \begin{bmatrix} 1 & 2 & & i-1 & i & n & & n \\ 0 & t_1 & \cdots & t_{i-2} & t_{i-1} & t_i & \cdots & t_{n-1} \end{bmatrix}; \tag{31}$$

by successive application of (31) we prove

$$\begin{bmatrix} n & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix} \approxeq \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix}. \tag{32}$$

Analogously

$$\begin{bmatrix} 1 & 2 & & n-1 & n \\ 0 & t_1 & \cdots & t_{n-2} & t_{n-1} \end{bmatrix} \approxeq \begin{bmatrix} 1 & 2 & & n-1 & n \\ 0 & t_1 & \cdots & t_{n-2} & n-1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & & n-1 & n-1 \\ 0 & t_1 & \cdots & t_{n-2} & t_{n-1} \end{bmatrix};$$

but the second term is equivalent to zero, since it does not depend on the nth component of the vertices $q_1, q_2, \cdots, q_{n+1}$. Hence

$$\begin{bmatrix} 1 & 2 & & n-1 & n \\ 0 & t_1 & \cdots & t_{n-2} & t_{n-1} \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & & n-1 & n \\ 0 & \mathrm{t} & \cdots & t_{n-2} & n-1 \end{bmatrix}$$

and more generally

$$(33) \quad \begin{bmatrix} 1 & 2 & & i-1 & i & & i+1 & n \\ 0 & t_1 & \cdots & t_{i-2} & t_{i-1} & i & \cdots & n-1 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & & i-1 & i & i+1 & n \\ 0 & t_1 & \cdots & t_{i-2} & i-1 & i & n-1 \end{bmatrix};$$

successive application of formula (33) yields

$$(34) \quad \begin{bmatrix} 1 & 2 & 3 & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & & n \\ 0 & 1 & 2 & \cdots & n-1 \end{bmatrix}.$$

From (32) and (34) it follows that

$$(35) \quad \begin{aligned} f_n^\epsilon(q_1, q_2, \cdots, q_{n+1}) &= \begin{bmatrix} 1 & n & n & & n \\ 0 & t_1 & t_2 & \cdots & t_{n-1} \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & & n \\ 0 & 1 & 2 & \cdots & n-1 \end{bmatrix} \\ &= \prod_{i=1}^{n} \int_{i-1}^{i} d\phi_1[t_i, q_i, q_{i+1}, T_i]. \end{aligned}$$

Write

$$(36) \quad F_i^\epsilon(q_i, q_{i+1}) = \int_{i-1}^{i} d\phi_1[t, q_i, q_{i+1}, T_i].$$

F_i^ϵ defines on t_n a one-dimensional ∇-cycle U.

If t_1^i is the equator of t_n, defined by the formulas

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{i-1} = \lambda_{i+1} = \cdots = \lambda_n = 0,$$

then $U_1^i \times t_1^i = \alpha$, the intersection index being taken in t_n. This follows from the same arguments as those which proved formula (28).

It is clear from (35) that the ∇-cycle Z^n defined by the functions f_n^ϵ on t_n is homologous to the Alexander product of the cycles $U_1^1 \times U_1^2 \times \cdots \times U_1^n$.

We need to prove that on t_n

$$U_1^1 \times U_1^2 \times \cdots \times U_1^n = \alpha^n.$$

To this end we note that if $q_1(\lambda_1, 0, \cdots, 0)$ and $q_2(\lambda_1', 0, \cdots, 0)$ are two close parametrizations of t_1', then $F_1^\epsilon(q_1, q_2)$ [see (36)] denotes the T_1-area of the sphere S_2 between the meridians (λ_1) and (λ_1'). This may be considered as an element of length of the equator t_1', corresponding to its small one-dimensional simplex (q_1, q_2). If we turn from ∇-cycles modulo the reals to de Rham differentials, then the ∇-cycle U_1^1 corresponds to a differential, namely the element of length on the equator t_1^1, and more generally the ∇-cycle U_1^i corresponds to a differential, namely the element of length on the equator t_1^i of the torus t_n. The expression for the Alexander product of the cycles $U_1^1, U_1^2, \cdots, U_1^n$ shows that the function defining this product expresses the volume element of the torus t_n, or rather of the n-dimensional parallelepiped from which t_n was obtained by identification of opposite faces. The volume of t_n is equal to the product of the lengths constituting $t_1^1, t_1^2, \cdots, t_1^n$, i.e., to α^n. Therefore the intersection index on t_n of the product of the cycles $(U_1^1 \times U_1^2 \times \cdots \times U_1^n)$ with t_n is equal to α^n. On t_n this product is ∇-homologous to the ∇-cycle Z^n. Hence $Z^n \times t_n = \alpha^n \neq 0$.

Thus we have proved that both the cycles Z^n with the reals as coefficient group, and the Δ-cycles t_n, are not homologous to zero.

From arguments analogous to those given in the proof of Lemmas 6 and 7 of §17, we obtain the following.

Every n-dimensional Δ-cycle in R with real coefficients is homologous to γt_n, where γ is a real number. Every n-dimensional ∇-cycle is homologous to γZ^n,

$$Z^n \times Z^m \overset{\infty}{\nabla} C_{n,m} Z^{n+m},$$

$$t_n \times Z^n = \alpha^n \neq 0.$$

These equations characterize the homology groups and the ring of products in R with real coefficients.

22. The n-dimensional case.

Space of arcs on the n-dimensional sphere. The methods and results of §§15—21 extend to the case of the more general space $R = R(S_n, a, b)$ of all rectifiable arcs on the n-dimensional sphere joining the points a and b on it.

Denote by Z_c^{n-1} the $(n-1)$-dimensional ∇-cycle in $R(S_n, a, b)$ defined as follows. On the n curves

$$q_1, q_2, \cdots, q_n, \quad \rho(q_i, q_j) < \epsilon, \qquad i, j = 1, 2, \cdots, n,$$

we stretch an n-dimensional film. For example, we introduce a parameter t on the curves q_i so that the points a_i^t with common values of t are distant from one another by distances not exceeding ϵ. On the points $a_1^t, a_2^t, \cdots, a_n^t$ we stretch an $(n-1)$-dimensional spherical simplex. The collection of all these simplexes for $0 \leqq t \leqq t_1$ forms an n-dimensional film. We denote the degree of covering of the point c by this film by

$$\psi_1(t_1; q_1, q_2, \cdots, q_n; c).$$

The functions

$$f_1^c(q_1, q_2, \cdots, q_n) = \psi_1(1; q_1, q_2, \cdots, q_n; c)$$

define an $(n-1)$-dimensional ∇-cycle Z_c^{n-1} modulo any prime p.

Further, define an $m(n-1)$-dimensional ∇-cycle as follows. Suppose that

$$\begin{aligned} &\psi_m(q_1, q_2, \cdots, q_n, q_{n+1}, \cdots, q_{m(n-1)+1}; c_1, c_2, \cdots, c_m) \\ &\quad = \int_0^1 d\psi_1(t_1; q_1, q_2, \cdots, q_n; c_1) \int_{t_1}^1 d\psi_1(t_2; q_n, q_{n+1}, \cdots, q_{2n-1}; c_2) \cdots \\ &\qquad\qquad \cdots \int_{t_{m-1}}^1 d\psi_1(t_n; q_{(m-1)(n-1)+1}, \cdots, q_{m(n-1)+1}; c_m). \end{aligned}$$

The functions

$$\psi_m(q_1, q_2, \cdots, q_{m(n-1)+1}; c_1, c_2, \cdots, c_m)$$

define the $m(n-1)$-dimensional ∇-cycle $Z_{c_1 c_2 \cdots c_m}^{m(n-1)}$. Denote by $Z^{m(n-1)}$ a ∇-cycle, ∇-homologous to any $Z_{c_1 c_2 \cdots c_m}^{m(n-1)}$. We have

$$Z^{k(n-1)} \times Z^{l(n-1)} \overset{\infty}{\nabla} C_{k,l} Z^{(k+l)(n-1)},$$

where $C_{k,l}$ is defined as in §16.

For $r = m(n-1)$ all the r-dimensional ∇-cycles are ∇-homologous to qZ^r. For $r \neq m(n-1)$ all the r-dimensional ∇-cycles are ∇-homologous to zero.

If a and b are diametrically opposed points of the sphere S_n, then the collection of meridians—semicircles joining a and b—is homeomorphic to $(n-1)$-dimensional projective space. If γ is a point of projective $(n-1)$-space, we denote by (γ) the meridian corresponding to γ under the mapping of the set of meridians onto this projective space. We denote by $(\gamma_1, \gamma_2, \cdots, \gamma_n)$ the polymeridian formed by the sum of n meridians $(\gamma_1) - (\gamma_2) + \cdots + (-1)^n(\gamma_n)$. The collection of polymeridians $(\gamma_1, \gamma_2, \cdots, \gamma_{2k+1})$ forms an $(n-1)(2k+1)$-dimensional Δ-cycle in R, which we shall denote by $t_{(n-1)(2k+1)}$. The collection of polymeridians $(\gamma_1, \gamma_2, \cdots, \gamma_{2k}, \overset{0}{\gamma}_{2k+1})$ for fixed $\overset{0}{\gamma}_{2k+1}$ forms a $2k(n-1)$-dimensional ∇-cycle $t_{2k(n-1)}$.

We have

$$t_{m(n-1)} \times Z^{m(n-1)} = \pm 1.$$

All r-dimensional Δ-cycles for $r = m(n-1)$ are Δ-homologous to qt_r. For $r \neq m(n-1)$ all r-dimensional Δ-cycles are homologous to zero.

Denote by $I(q)$ the length of an arc in S_n in the Riemannian metric. Suppose that (z_n) is the collection of Δ-cycles for which

$$z_r \times Z^r \neq 0.$$

Let

$$c_l = \inf_{z_{l(n-1)} \in (z_{l(n-1)})} \left[\sup_{q \in z_{l(n-1)}} I(q) \right].$$

There exists a geodesic arc of length c_l joining arbitrary points a and b on the sphere S_n with any Riemannian metric on it or on any Riemannian manifold homeomorphic to S_n.

If $c_{2k} = c_{2k+1} = c$, then there exists an $(n-1)$-dimensional Δ-cycle not homologous to zero, consisting of geodesic arcs joining a and b. In this case all the geodesics passing through a pass also through b, and the lengths of these arcs between a and b are equal to c.

Finally, the merging $c_l = c_{l+2}$ is impossible. Always $c_l < c_{l+2}$.

The numbers c_l increase with l as an arithmetic progression.

CHAPTER V

Closed Geodesics on Manifolds Homeomorphic to the n-Dimensional Sphere

Among the problems of the calculus of variations in the large the greatest attention, beginning with the well-known work of Poincaré, has been paid to the investigation of closed geodesics on manifolds homeomorphic to the n-dimensional sphere. In the two-dimensional case the question was completely resolved in the joint paper of the author and L. G. Šnirel′man [**19**]. In fact it was proved that for a surface of genus zero there exist three closed non-selfintersecting geodesics. The proof was based on the investigation of the invariant of category and the special deformations referred to in Chapter III. The n-dimensional case was first investigated by Morse and his school. Indeed, he proved that on an n-dimensional manifold R_n homeomorphic to the sphere S_n there are generally speaking $n(n+1)/2$ closed geodesics. In order to guarantee that each of them is not a twice or more repeated arc, Morse introduces metric restrictions: it is possible to map R_n onto S_n in such a way that the ratio between corresponding linear elements lies between m and $M < 2m$. But even under these restrictions all of these geodesics may *a priori* merge. We shall prove that on each manifold homeomorphic to the sphere S_n there are either $n+1$ geodesics of equal length or else a continuous family of geodesics. For $n = 2$, using the special transformations mentioned above, we obtain a new proof of the theorem on three non-selfintersecting geodesics. In the general case ($n > 2$) the Morse restriction continues to be needed in order to verify that these geodesics do not reduce to the repetition of a single geodesic.

The proof is based on the investigation of certain ∇-cycles and their products in the space $P(S_n)$ of rectifiable closed curves S_n, and on the application of the fundamental theorem.

23. **The case** $n = 2$. On the sphere S_2 we choose a point a. Suppose that q_1 and q_2 are two curves of $P = P(S_2)$, where $\rho(q_1, q_2) < \epsilon < n$. We denote by $f_1^\epsilon(q_1, q_2)$ the order modulo 2 of circuiting of the point a by the curve $q_1 - q_2$ or the degree of covering of a by a film stretched on q_1 and q_2. We must choose on q_1 and q_2 a cyclic parameter ϕ, so that

the distance between the points $a^{\phi}_{q_1}$ and $a^{\phi}_{q_2}$ of the curves q_1 and q_2 with common values of the parameter ϕ is less than ϵ. We join all these pairs of points by geodesic arcs of length $< \epsilon$, forming a film whose degree of covering of the point a is also $f_1^{\epsilon}(q_1, q_2)$. The family of functions $f_1(q_1, q_2)$ as $\epsilon \to 0$ defines a one-dimensional ∇-complex, which is a ∇-cycle mod 2. We denote it by Z_1^a.

All the Z_1^a are ∇-homologous to one another. We denote by $\phi(q_1)$ the intersection index of the curve q_1 with an arc joining a and b. Then

$$f_1^{\epsilon}(q_1, q_2, a) - f(q_1, q_2, b) = \phi(q_1) - \phi(q_2). \tag{1}$$

$\phi(q_1)$ defines a zero-dimensional ∇-cycle Z_0. Equation (1) shows that $Z_1^a - Z_1^b$ is the $\overset{\infty}{\nabla}$-boundary of Z_0.

Therefore $Z_1^a \overset{\infty}{\nabla} Z_1^b$. We denote the common homology class of all the cycles Z_1^a by Z_1.

Further, we write

$$Z^2 = Z^1 \times Z^1; \quad Z^3 = Z^2 \times Z^1.$$

We have classes $[Z^1]$, $[Z^2]$, $[Z^3]$ of one, two, and three-dimensional ∇-cycles. Denote by z_3 the three-dimensional cycle consisting of all circles. We have

$$Z^3 \times z_3 = 1.$$

In fact, the set z_3 of all circles on the sphere is a three-dimensional projective space, given that one identifies all the circles of z_3 degenerating into a point. Denote by z_1^{ab} the set of circles of z_3 passing through the points a and b. z_1^{ab} is a projective line in Z_3, Z_1^c induces on the projective space z_3 a one-dimensional ∇-cycle $\overline{Z}_1^c$, whose intersection index in z_3 with z_1^{ab} is equal to 1, i.e., $\overline{Z}_1^c$ is a one-dimensional ∇-cycle in projective space dual to the projective line. A one-dimensional Δ-cycle dual to the projective line is equivalent to the projective plane, so that

$$\overline{Z}_1^a \times \overline{Z}_1^b \times \overline{Z}_1^c = 1.$$

(The intersection index of three projective planes in projective space is equal to 1.)

Hence

$$Z^3 \times z_3 = (Z_1^a \times Z_1^b \times Z_1^c) \times z_3 = 1.$$

We shall call K-sets closed sets of non-selfintersecting curves which are results of normal deformation of a subset of the collection of circles on the sphere S.

We denote by $[z_1]$, $[z_2]$, $[z_3]$ classes of Δ-cycles such that for

$$z_i \subset [z_i],\ z_i \times Z^i = 1;\ (i = 1,2,3).$$

All the classes $[z_i]$ are nonempty. For example, the class $[z_3]$ contains z_3 [see (3)]. Here the class $[z_1]$ is subordinated to the class $[z_2]$ by means of Z^1 and to the class $[z_3]$ by means of Z^2. The class $[z_2]$ is subordinated to the class $[z_3]$ by means of Z^1.

We denote by $c_i (i = 1,2,3)$ the lower limit of the maxima of the lengths of the curves q on the sets of cycles of the classes $[z_i]$:

$$c_i = \inf_{z_i \in [z_i]} \sup_{q \in z_i} I(q).$$

We have $c_1 \leqq c_2 \leqq c_3$.

Further, $c_1 > 0$. In fact, the family of curves of length less than h may be reduced to points.

If z_1' is a cycle of class $[z_1]$ on which the maximum of $I(q)$ is less than h, then z_1' may be deformed into a one-dimensional cycle of curves degenerating into points (more simply into a set of points, a set of dimension 1 not covering the sphere S_2). At some moment of the deformation sufficiently close to the final moment, the curves of z' go into curves of very small diameter, but still distinct from points, and their family z'' still does not cover S_2. $z'' \in [z_1]$, $Z_1^a \in Z^1$. But $z'' \times Z_1^a = 1$, so that for any point a $z'' \times Z_1^a = 1$ (even though z'' does not cover a), which is impossible, since the set-theoretical intersection of z'' and the support Z_1^a is empty. Thus there do not exist cycles of $[z_1]$ in the region $(I \leqq h)$, so that

$$c_1 \geqq h.$$

The level surfaces $(I = c_1)$, $(I = c_2)$, $(I = c_3)$ contain closed non-selfintersecting geodesic curves of length c_1, c_2, c_3, since on these level surfaces there exist nonempty K-critical sets, and K-critical sets consist of such geodesics, by Theorem 2 of §13.

If $c_1 = c_2 = c$ or $c_2 = c_3 = c$, then the level surface $(I = c)$ must contain a one-dimensional Δ-cycle, included in the critical set of the class $[z_1]$. This follows from the fundamental theorem, applied to the classes $[z_1]$ and $[z_2]$ and to $[z_2]$ and $[z_3]$ respectively, the first of which is subordinated to the second by means of the class $[Z_1]$. But then the family of closed non-selfintersecting geodesics of length c covers S_2, for otherwise it could not contain cycles nonhomologous to zero.

If $c_1 = c_2 = c_3 = c$, then the level surface $(I = c)$ contains a two-dimensional cycle z_2' such that $z_2' \in [z_2']$, $z_2' \overset{\infty}{\nabla} 0$, while z_2' is contained in the K-critical set, which means that z_2' is contained in the set of closed non-selfintersecting geodesics of length c_1. This means that there exists a two-parameter family of closed non-selfintersecting curves of length c. Through each pair a and b of points of the sphere there must pass such a non-selfintersecting closed geodesic, since for a cycle z_2' consisting of such geodesics $z_2' \times (Z_1^a \times Z_1^b) = 1$. This follows from the application of the fundamental theorem to the classes $[z_1]$ and $[z_3]$, where $[z_1]$ is subordinated to $[z_3]$ by means of Z^3.

It is not difficult to verify that in this case all the geodesics are closed nonintersecting curves of length c. Thus we have arrived at a theorem proved by other means in our paper with L. G. Šnirel′man.

On a three-times differentiable surface of genus zero there exist either three closed non-selfintersecting geodesics of different lengths, or else a family of such geodesics of equal length and covering S_2 and one such geodesic of different length, or else all the geodesics are closed non-selfintersecting curves of the same length.

24. **The case** $n > 2$. We turn to the investigation of closed geodesics on manifolds homeomorphic to the n-dimensional sphere ($n > 2$). For this we need to investigate certain properties of the ring of products of the space $P = P(S_n)$ of rectifiable closed curves on the n-dimensional sphere S_n. Hence it follows that on a manifold homeomorphic to S_n there are either closed geodesics of $(n + 1)$ distinct lengths, or else a continuous system of such geodesics. This generalizes the theorem proved above for the two-dimensional sphere S_2. Under the metric restrictions of Morse the corresponding geodesics do not reduce to a repetition of one of them.

We shall consider in detail the case $n = 3$. In the space $P = P(S_3)$ we select two special sets: the set $\mathfrak{A}$ of all curves reducing to points, and the set $\mathfrak{B}$ of all curves reducible to a two or more times repetition of a simple arc. One may metrize P so that $\mathfrak{A}$ and $\mathfrak{B}$ become infinitely separated elements. A finite set in P is a set lying outside some neighborhood of $\mathfrak{A} + \mathfrak{B}$. An $\mathfrak{A}$- or $\mathfrak{B}$-finite set is one which lies outside some neighborhood of $\mathfrak{A}$ or, respectively, of $\mathfrak{B}$.

We suppose that S_3 lies in Euclidean 4-space E_4. The family A_6 of all circumferences on S_3 is a 6-dimensional $\mathfrak{B}$-finite Δ-cycle. Suppose that A_5^f is the collection of all circumferences whose planes in E_4 are orthogonal to the plane f, i.e., contain a line orthogonal to f. Let A_4 be the collection of all circumferences passing through the point a. A_5^f and A_4^f are 5- and 4-

dimensional $\mathfrak{B}$-finite Δ-cycles on P and at the same time in A_6. We shall take our homologies mod 2.

In A_6 we have

$$A_4^a \times A_4^b \times A_5^f \times A_5^g = 1. \tag{2}$$

For the proof we choose in the coordinate space $E_4(x, y, z, u)$ a plane f defined by the equations $x = y = 0$, and a plane g defined by the equations $y = u = 0$. Suppose that $a(0,0,1,0)$ and $b(0,0,0,1)$ are selected points of the unit sphere S_3.

A circumference r lying in the plane $x = y = 0$ is a set-theoretical intersection

$$A_5^f A_5^g A_4^a A_4^b.$$

In fact, the plane p of a circumference lying in $A_4^a A_4^b A_5^f A_5^g$ passes through a and b. Since p is orthogonal to f, it contains a line l orthogonal to f, i.e., orthogonal to the two lines

$$y = z = u = 0, \quad x = y = u = 0.$$

The direction cosines of l are proportional to

$$0 : k_1 : 0 : k_2. \tag{3}$$

Analogously the plane p contains the line l orthogonal to g with direction cosines orthogonal to those of l:

$$k_3 : 0 : k_4 : 0. \tag{4}$$

Two cases are possible.

Case 1. *p contains a point c of the type* $(x, y, 0, 0)$. Then we find that for each $c(x^0, y^0, 0, 0)$ the plane p passing through a, b and c contains lines l and l_1 with direction cosines expressed by (3) and (4). We pass the lines l and l_1 throuch c. These lines are not parallel to the line l_2 joining a and b, and they intersect l_2 at the points d and d_1. The parametric equation of l_2 may be taken to be

$$x = y = 0, \quad z = t, \quad u = 1 - t. \tag{5}$$

Suppose that $d = d(0, 0, t_0, 1 - t_0)$ and $d_1 = d_1(0, 0, t_1, 1 - t_1)$. The direction cosines of l, joining c and d, are proportional to the differences of the coordinates of c and d, i.e., to

$$x_0 : y_0 : (-t_0) : (t_0 - 1). \tag{6}$$

From (6) and (3) $x^0 = 0$. Analogously, the direction cosines of l_1 are proportional to

$$x^0 : y^0 : (-t_1) : (t_1 - 1). \tag{7}$$

It follows from (7) and (4) that $y^0 = 0$.

Thus, in Case 1 the plane p passes through the origin of coordinates and its equation is $x = y = 0$.

Case 2. *The plane p contains no point c of the form $c(x_0, y_0, 0, 0)$.* Then the projection of p on the plane $x = y = 0$ coincides with the line l_2 joining a and b and consisting of points of the form $(0, 0, t, 1 - t)$. If the point d' lies on p off l_2, then $d' = d'(x', y', t', 1 - t')$, where $|x'| + |y'| > 0$. By hypothesis the plane p contains a line l_3 orthogonal to f. Such a line may be passed through some point $d'(x', y', t', 1 - t')$. It intersects l_2 at some point $d''(x'', y'', t'', 1 - t'')$. The direction cosines of l_3 are proportional to

$$x' : y' : (t' - t'') : (t'' - t'). \tag{8}$$

Comparing (8) and (2) we obtain $x' = 0$. Analogously, passing a line l_4 orthogonal to g through d and comparing (8) and (4), we obtain $y' = 0$. But this means that any point d chosen on p lies on the line l_2. Thus the plane p does not exist.

Thus the set-theoretical intersection $A_4^a A_4^l A_4^k A_5^g$ consists of the single element r, lying in the plane $x = y = 0$. For the proof of (2) we need to establish that the intersecting cycles lie in the neighborhood of r in general position. The plane of the circumference r passes through the points $a(0, 0, 1, 0)$, $b(0, 0, 0, 1)$, $c(0, 0, 0, 0)$.

The planes of circumferences close to r pass through the points

$$a'(x_1, y_1, 1, 0), \quad b'(x_2, y_2, 0, 1), \quad c'(x_3, y_3, 0, 0),$$

where $x_1, y_1, x_2, y_2, x_3, y_3$ are close to zero.

In the neighborhood of r the numbers $x_1, y_1, x_2, y_2, x_3, y_3$ may be considered as 6 coordinates of a plane close to the plane of r (or of the corresponding circumference lying in that plane). A_4^a is defined in the neighborhood of t_r by the equations

$$x_1 = 0, \quad y_1 = 0 \tag{9}$$

and A_4^b by the equations

$$x_2 = 0, \quad y_2 = 0. \tag{10}$$

A_5^f, as follows from the foregoing, is defined by the equation

$$x_3 = 0, \tag{11}$$

and A_5^g by the equation

$$(12) \qquad y_3 = 0.$$

Since the determinant of the system (9)—(12) is equal to 1, the intersecting cycles lie in r in general position, and (2) is proved.

Write $A_2 = A_2^{afg} = A_4^a \times A_5^f \times A_5^g$. It follows from (2) that $A_2 \times A_4^b = 1$. The family of circumferences A_2 covers the sphere S_3 (since A_2 has an element in common with each A_4^b, i.e., there is in A_2 a circumference passing through any point of S_3).

∇-cycles in P. We turn to the consideration of ∇-cycles in P. Suppose that q_1, q_2, q_3 are three finite elements of P_1, $\rho(q_1, q_2) < \epsilon$, $\rho(q_2, q_3) < \epsilon$, $\rho(q_1, q_3) < \epsilon$, where ϵ is positive and $\leqq 1/2$. On the close curves q_1, q_2, q_3 one may choose cyclic parameters ϕ such that the points $a_\phi^1, a_\phi^2, a_\phi^3$ of these curves corresponding to the same value of ϕ are distant from one another by a distance $\leqq \epsilon$. On each pair of curves q_1, q_2; q_1, q_3; q_2, q_3 one may stretch a film whose boundary will be correspondingly $q_1 + q_2$; $q_1 + q_3$; $q_2 + q_3$ (for example, one may join by arcs the pairs of points a_ϕ^1, a_ϕ^2; a_ϕ^1, a_ϕ^3; a_ϕ^2, a_ϕ^3, corresponding to a common value of the parameter ϕ, such that these arcs vary continuously with ϕ and have lengths not exceeding ϵ). The sum of these films stretched on q_1, q_2, q_3 forms a closed surface. The linking index of this surface with the point a is denoted by

$$f_1^\epsilon(q_1, q_2, q_3; a).$$

The functions $f_1^\epsilon(q_1, q_2, q_3; a)$ define a two-dimensional ∇-cycle, which we shall denote by Z_2^a. We may show that this cycle does not depend, up to homology, on the choice of the film.

Remark. We could have defined three-dimensional tubes stretched on the curves q_1, q_2, q_3, whose boundaries would have been the films defined above, and then defined the functions f_2^ϵ as the degrees of covering of the point a by these tubes.

Further we define a one-dimensional ∇-cycle Z_1^f in the following way. For the pair of finite curves q_1, q_2 with $\rho(q_1, q_2) < \epsilon$, we define a common direction of circuiting. We consider the projections of these curves on the plane f. If these projections have areas of the same sign, put

$$f_1^\epsilon(q_1, q_2, q_3; f) = 0,$$

and if they are different put $f_1^\epsilon(q_1, q_2, q_3; f) = 1$. The functions f_1^ϵ define a one-dimensional ∇-cycle Z_1^f.

The support of the ∇-cycle Z_2^a is the set of all finite curves passing through the point a, and the support of the ∇-cycle Z_1^f the set of finite curves whose projections on f have area zero.

LEMMA 1.

$$(Z_2^a \times Z_2^b \times Z_1^f \times Z_1^g) \times A_6 = 1.$$

On the manifold A_6 the ∇-cycles $Z_2^a, Z_2^b, Z_1^f, Z_1^g$ induce Δ-cycles of the same dimension:

$$\overline{Z_2^a}, \overline{Z_2^b}, \overline{Z_1^f}, \overline{Z_1^g}.$$

On A_i the tubes stretched on the circles q_1, q_2, q_3, $\rho(q_i, q_j) < t$ $(i, j = 1, 2, 3)$, will be constructed as follows. Form the family $Q(q_1, q_2, q_3)$ of all circles $q \in A_i$, forming in A_6 a three-dimensional simplex with vertices q_1, q_2, q_3. The degree of covering by this tube of the point a is denoted by $f_2^\epsilon(q_1, q_2, q_3; a)$. The function $f_2^\epsilon(q_1, q_2, q_3; a)$ defines in A_6 a two-dimensional ∇-cycle Z_2^a.

The two-dimensional ∇-cycle Z_2^a is equivalent on A_6 to the four-dimensional Δ-cycle A_4^a. Indeed, suppose that the two-dimensional ϵ-simplexes (q_1, q_2, q_3) are in general position relative to the 4-dimensional manifold A_4^a. The intersection index of this simplex, for sufficiently small $\epsilon > 0$, is equal to 1 or 0 according as this simplex contains the "points" A_4^a or not. In the first case the tube $Q(q_1, q_2, q_3)$ contains a single circumference passing through the point a, and itself contains the point a, and the degree of its covering of the point a is equal to 1. In the second, the degree of covering of the point a by this tube is equal to zero. Thus, this degree of covering, expressed by the function $f_2^\epsilon(q_1, q_2, q_3; a)$, coincides with the intersection index of the ϵ-simplex (q_1, q_2, q_3) with the Δ-cycle A_4^a.

The ∇-cycle $\overline{Z_2^a}$ is defined in A_6 by a function of a two-dimensional simplex, equal to the intersection index of this simplex with a Δ-cycle A_4^a of complementary dimension, i.e., the ∇-cycle $\overline{Z_2^a}$ is equivalent on A_4 to the Δ-cycle A_4^a.

Analogously, the ∇-cycle $\overline{Z_1^f}$ is equivalent on A_6 to the Δ-cycle A_5^f. The intersection index of the one-dimensional ϵ-simplex (q_1, q_2) with A_5^f is equal to 1 or zero depending on whether this element has common elements with A_5^f or not. Here ϵ is sufficiently small, and (q_1, q_2) is in general position relative to A_5^f. A_5^f consists of circumferences whose planes are orthogonal to f. The ϵ-simplex (q_1, q_2) consists of a one-parameter family of circumferences on which for sufficiently small ϵ there has been chosen continuously an order of circuit. If (q_1, q_2) does not intersect A_5^f, the projections of these circumferences on f have the same order of circuit. These projections are in fact ellipses which do not degenerate into line segments. Then $f_1^{\prime}(q_1, q_2, q_3; f)$ is by definition equal to zero. If (q_1, q_2) intersects A_5^f, the corresponding projections, at one point turning into segments, change the order of circuiting and the sign of the area, and $f_1^{\prime}(q_1, q_2; f) = 1$. Thus, the functions of simplexes (q_1, q_2) defining the cycle $\overline{Z}_1^f$ on A_6, coincide with the intersection index of this simplex with the Δ-cycle A_5^f. Thus, $\overline{Z}_1^f$ is equivalent on A_6 to the Δ-cycle A_5^f.

Therefore on A_6 $(\overline{Z_2^a} \times \overline{Z_2^b} \times \overline{Z_1^f} \times \overline{Z_1^g})$ coincides with the intersection index on A_6,

$$(A_4^a \times A_4^b \times A_5^f \times A_5^g) = 1.$$

But $(Z_2^a \times Z_2^b \times Z_1^f \times Z_1^g) \times A_6$ in P coincides with the intersection index $\overline{Z_2^a} \times \overline{Z_2^b} \times \overline{Z_1^f} \times \overline{Z_1^g}$ in A_6 and is therefore equal to 1.

Multiplication of ∇-cycles. Consider the following cycles:

$$Z_{abfg}^6 = Z_2^a \times Z_2^b \times Z_1^f \times Z_1^g,$$

$$Z_{abf}^5 = Z_2^a \times Z_2^b \times Z_1^f,$$

$$Z_{ab}^4 = Z_2^a \times Z_2^b,$$

$$Z_{afg}^4 = Z_2^a \times Z_1^f \times Z_1^g,$$

$$Z_{af}^3 = Z_2^a \times Z_1^f,$$

$$Z_a^2 = Z_2^a,$$

$$Z_{fg}^2 = Z_1^f \times Z_1^g,$$

$$Z_f^1 = Z_1^f.$$

Denote by (Z^6) the class of six-dimensional ∇-cycles homologous to Z_{abfg}^6, by (Z^5) all the five-dimensional cycles ∇-homologous to Z_{abf}^5, by (Z_{ab}^4) the four-dimensional cycles ∇-homologous to Z_{ab}^4, Z_4' the four-dimensional cycles ∇-homologous to Z_{afg}^4, by (Z^3) the three-dimensional cycles ∇-homologous to Z_{af}^3, by (Z^2) the two-dimensional cycles ∇-homologous to Z_a^2, by (Z_2') the two-dimensional cycles ∇-homologous to $Z_{fg}'^2$, and finally by (Z^1) the one-dimensional cycles ∇-homologous to (Z_1^f).

Lower K-cycles. We shall consider K-sets in P, defined in the sense of the remark to Theorem 2, §13, Chapter III. K-deformations of a family of circumferences are normal deformations of them, leaving invariant single points and circumferences of sufficiently small radius, and never in the course of the deformation transforming any circumference into a multiple repetition of a simple arc. In other words, K-deformations are normal deformations changing only the finite portion of a subset A, that finite portion of A throughout the deformation remaining in a finite portion of P.

K-sets in P are the results of K-deformations of a certain subset A_6 (family of circumferences), and correspondingly K-cycles are the results of K-deformations of a Δ-cycle of A_6. Since K-deformations of a Δ-cycle take place in a finite portion of P, they do not change its intersection indices with ∇-cycles.

(z_6), (z_5), (z_4), (z'_4), (z_3), (z_2), (z'_2), (z_1) are classes of Δ-cycles whose intersection indices with cycles of the corresponding classes (Z^6), (Z^5), (Z^4), (Z'^4), (Z^3), (Z^2), (Z'^2), (Z^1) are equal to 1. All of these classes are nonempty. Thus A_6 lies in (z_6), $A_5^f \in (z_5)$, $A_4^a \in (z_4)$, $A_4^{fg} \in (z'_4)$, $A_3^{af} \in (z_3)$, $A_2^{afg} \in (z_2)$. In addition, these classes are subordinated. In fact, (z_2) is subordinated to (z_3), (z_3) is subordinated to (z_4), (z'_4) is subordinated to (z_5), (z_5) is subordinated to (z_6) by means of the class Z^1. Further, (z_2) is subordinated to (z_4), (z_3) is subordinated to (z_5), (z_4) is subordinated to (z_6) by means of (Z^2); (z'_4) is subordinated to (z_6) by means of (Z'_2); (z_2) is subordinated to (z_5), (z_3) is subordinated to (z_6) by means of (Z^3); (z_2) is subordinated to (z_6) by means of (Z^3); (z_2) is subordinated to (z_6) by means of (Z^4).

A sketch of the schema of subordination of these classes, with the subordination indicated by an arrow, is as follows:

$$\begin{tikzcd}
 & z_6 \arrow[dl] \arrow[dr] & \\
z'_4 \arrow[d] & & z_5 \arrow[d] \\
z_3 \arrow[dr] & & z'_4 \arrow[dl] \\
 & z_2 &
\end{tikzcd}$$

The cycles (z_1) are finitely nonhomologous to zero, i.e., they cannot be the boundaries of finite two-dimensional Δ-complexes. An example of such a cycle of (z_1) could be the set z_1 of circumferences on a two-dimensional sphere S_2, lying in S_3, and passing through two points a and b. This cycle may be deformed by an infinite deformation into an infinitely distant element, a point or a twice repeated simple arc. An example of a cycle of the class (z'_2) could be the collection of all great circles on the sphere S_2. Cycles of class (z_2') can be brought into the infinite element p by an infinite deformation, but they are not homologous to zero in the finite part of P.

The functional $I(q)$ on P. Each hypersurface homeomorphic to a sphere S_3 may be considered as a sphere S_3 on which there is given some metric. We shall denote by $I(q)$ the length of a curve q of P in the natural metric.

Suppose that $I(q)$ is a functional denoting the length of the arc q. Denote by $c_i\,(i = 1,2,3,4,5,6)$ and by $c_j'\,(j = 2,4)$ the lower bounds of the maxima of $I(q)$ on cycles of the classes (z_i) and (z_j'). c_1 and c_2' are equal to zero.

We shall show that c_2, c_3, c_4, c_4', c_5, c_6 are not equal to zero. Since $c_4' \geqq c_2$, $c_{2+k} \geqq c_2$ for $k > 0$, then it suffices to prove that $c_2 > 0$.

If $c_2 = 0$ there exists a K-deformation carrying a cycle $z_2 \in (z_2)$ and lying in A_6 into a cycle $\bar{z}_2$ on which the maximum length of curves is less than $h/4$, h being any number less than the diameter of the sphere. We shall show that this is impossible.

We introduce the concept of a normal two-dimensional ϵ-simplex (q_1, q_2, q_3) in P. Select a cyclic parameter ϕ on q_1, q_2, q_3 such that the points a_1^ϕ, a_2^ϕ, a_3^ϕ are distant from one another by at most ϵ. Construct the spherical triangle with a_1^ϕ, a_2^ϕ, a_3^ϕ as vertices. Suppose that λ_1, λ_2, λ_3 are nonnegative numbers and that $\sum_{i=1}^{3} \lambda_i = 1$. The point $c(\phi, \lambda_1, \lambda_2, \lambda_3)$ is a point of the spherical triangle with the barycentric coordinates λ_1, λ_2, λ_3. When ϕ runs through the entire period of its values, the point $c(\phi, \lambda_1, \lambda_2, \lambda_3)$ for fixed λ_1, λ_2, λ_3 marks out a curve which we shall denote by $q(q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3)$. Obviously, for example, $q(q_1, q_2, q_3, 1, 0, 0) = q_1$. We shall say that $q(q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3)$ has barycentric coordinates λ_1, λ_2, λ_3 in the normal triangle (q_1, q_2, q_3) and so forth. The collection of curves $q(q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3)$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_3 \geqq 0$ is a normal simplex with the vertices q_1, q_2, q_3.

Suppose that $\mathscr{S}$ is a two-dimensional ϵ-complex, consisting of ϵ-simplexes, and suppose given a deformation D of all the vertices of these simplexes. At the moment t of this deformation the curves q_1, q_2, q_3 which are the vertices of the normal ϵ-simplex go into curves $D(q_i, t)$, $i = 1,2,3$. The points a_1^ϕ, a_2^ϕ, a_3^ϕ of the curves q_1, q_2, q_3, the distances between which do not exceed ϵ, go into points $D(a_i^\phi, t)$, $i = 1,2,3$ of the curves $D(q_i, t)$, while the distance between pairs of such points for any t does not exceed $\alpha > 0$, where α is arbitrarily small for sufficiently small ϵ. Suppose that $\alpha < h$. Consider the deformation D on the entire complex, and suppose that $q(q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3)$ is an element of the normal ϵ-simplex (q_1, q_2, q_3), i.e., the collection of points $c_\phi = c(\phi; \lambda_1, \lambda_2, \lambda_3)$ of the triangles a_1^ϕ, a_2^ϕ, a_3^ϕ. We shall show that $D(c_\phi, t)$ is a point of the spherical triangle with vertices $D(a_i^\phi, t)$, $i = 1, 2, 3$, and the same barycentric coordinates λ_1, λ_2, λ_3. The collection of points $D(c_\phi, t)$ for various values of ϕ forms a curve $D(q, t)$. The curves $D(q, t)$ fill out a normal ϵ-simplex with vertices $D(q_i, t)$, $i = 1, 2, 3$. Thus we have defined a normal deformation of the entire ϵ-simplex (q_1, q_2, q_3). We shall apply this construction to any ϵ-simplex and to our entire normal complex under the normal deformation of (q_1, q_2, q_3); a curve, an element with the barycentric coordinates λ_1, λ_2, λ_3 goes into a curve which is an element with the same barycentric coordinates in a simplex with vertices which are images of the initial vertices.

Consider a cycle $z_2 \subset A_c$ of class (z_2). Suppose that there exists a K-deformation D carrying it into some cycle $\bar{z}_2$ on which the maximum length of curves $q < h/4$. For each curve q' of $\bar{z}_2$ we find the center of gravity on the sphere S_2, $d(q') = d_{q'}$, and define a deformation D_1 carrying the curve q' into its center of gravity. A point c of the curve q' moves under the deformation D_1 along a geodesic arc $cd_{q'}$ with uniform velocity, and at the final moment of the deformation D_1 goes into $d_{q'} = d(q')$. The product $D_1 D$ is a deformation D_2, carrying z_2 into a collection of isolated points. Suppose that the final moment of the deformation D_2 corresponds to the value $t = 1$, and the initial moment to $t = 0$. At any moment $t \in (0, 1)$, $t < 1$ of the deformation D_2, the curve $q \in z_2$, distinct from a point, goes into the same curve. At any moment $t < 1$ the image z_2^t of the cycle z_2 must cover any point of the sphere S_2, since $z_2^i \times Z_a^2 = z^2 \times Z_a^2 = 1$.

Now we choose a number ϵ such that any curves of z_2 distant from one another by less than ϵ remain distant under the deformation D_2 by a distance less than $h/4$. We carry out a simplicial decomposition of z_2 into ϵ-simplexes (q_1, q_2, q_3). Define a deformation D' carrying each ϵ-simplex (q_1, q_2, q_3) of z_2 into a normal simplex with the same vertices (q_1, q_2, q_3). Write

$$\tilde{z}_2 = D' z_2.$$

At any moment t, $0 < t < 1$, the deformation D_2 carries the vertices of any ϵ-simplex (q_1, q_2, q_3) into curves $D_2(q_i, t)$, the distance between which $< \alpha < h/4$.

Define a deformation D^2 of the complex $\tilde{z}_2$ which for the vertices q_1, q_2, q_3 of simplexes of $\tilde{z}_2$ coincides with D_2 and then extends as a normal deformation onto the simplexes themselves, i.e., a curve which is an element q of the ϵ-simplex (q_1, q_2, q_3) with barycentric coordinates λ_1, λ_2, λ_3 goes into an element with the same barycentric coordinates of the simplex whose vertices are the images of q_1, q_2, q_3 at the corresponding moment t of the deformation.

At the moment $t = 1$ the vertices q_i $(i = 1, 2, 3)$ of each ϵ-simplex of $\tilde{z}_2$ go into points $a_i = D^2(q_i, 1)$. The curve $q(q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3)$ of the ϵ-simplex (q_1, q_2, q_3) with the barycentric coordinates λ_1, λ_2, λ_3 goes into the point $(\lambda_1, \lambda_2, \lambda_3)$ of the geodesic triangle a_1, a_2, a_3 with the barycentric coordinates λ_1, λ_2, λ_3. Thus the entire simplex (q_1, q_2, q_3) goes as a result of D_2 into a collection of points filling out a certain spherical triangle.

The entire cycle $\tilde{z}_2$ as a result of the deformation D^2 goes into a collection of points B, filling out a finite or countable number of geodesic triangles and therefore not covering the entire three-dimensional sphere S_2. At the moment $t_0 < 1$ of the deformation, sufficiently close to the final moment 1, the deformation carries the cycle $\tilde{z}_2$ into a set of curves of arbitrarily small length and filling out some arbitrarily small neighborhood of the set B and therefore not filling out the entire sphere S_3. We cut off the deformation D^2 at that moment t_0 and denote this truncated deformation by D^2_{t0}.

The product $D^2_{t0}D'$ is a deformation of the family of curves z_2, carrying it into the family $\bar{z}_2'$ which does not cover the sphere S_3. But there is no such deformation, since $\bar{z}_2' \times Z_2^a = 1$ for any $a \in S_3$. We have arrived at a contradiction, the escape from which consists in the inequality $c_2 > h/4 > 0$.

The numbers $c_i\,(i = 2, 3, 4, 5, 6)$, c_4' are k-critical values of $I(q)$.

There accordingly exist (see Theorem 2 of Chapter III) closed geodesics of lengths c_2, c_3, c_4, c_4', c_5, c_6. Thus generally speaking there are 6 closed geodesics.

In view of the rules for subordination of Δ-cycles of the classes (z_i) and (z_j) and the fundamental theorem we have:

If $c_2 = c_3 = c$; $c_3 = c_4 = c$; $c_4' = c_5 = c$; $c_5 = c_6 = c$, then there exists a one-dimensional Δ-cycle consisting of closed geodesics of length c and belonging to the class (z_1). Since all closed geodesics are situated in the finite portion of P, this cycle cannot be reduced to a single element.

If $c_2 = c_4' = c$; $c_3 = c_5 = c$; $c_4 = c_6 = c$, then there exists a cycle consisting of closed geodesics of length c, belonging to the class (z_2) and covering the entire hypersurface S_3.

If $c_4' = c_6 = c$, then there exists a cycle consisting of closed geodesics of length c in the class (z_2').

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If $c_2 = c_5 = c$; $c_3 = c_6 = c$, then there exists a cycle consisting of closed geodesics of length c and in the class (z_3).

If $c_2 = c = c_6$, then there exists a cycle of closed geodesics of length c of class (z_4). In this case all the geodesics are closed.

In every case either both groups of strong inequalities

$$c_2 < c_3 < c_4 < c_6, \qquad c_2 < c_4' < c_5 < c_6,$$

are satisfied, and then there exist at least 4 closed geodesics of distinct lengths, or else there exists a continuous family of closed geodesics.

As for the general n-dimensional case, let us suppose that $P = P(S_n)$ is the space of rectifiable curves on the n-dimensional sphere S_n. A_{3n-3} is a $(3n-3)$-dimensional cycle of all circumferences on S_n. Z_a^{n-1} is a $(n-1)$-dimensional ∇-cycle whose support is the collection of all curves of P passing through the point a. Z_f^1 is a one-dimensional ∇-cycle whose support is the family of all curves of P whose projections on the plane f have zero area. Repeating the preceding considerations, we obtain

$$(Z_{n-1}^a \times Z_{n-1}^b \times Z_1^{f_1} \times Z_1^{f_2} \times \cdots \times Z_1^{f_{n-1}}) \times A_{3n-3} = 1.$$

Thus a repetition of the preceding arguments will guarantee the existence of at least $(n+1)$ closed geodesics of different lengths or a continuous family of such geodesics on a manifold homeomorphic to the sphere S_n.

25. **The case** $n = 2^k$. For $n = 2^k$ one may sharpen the preceding result, by making use of the recent work of L. S. Pontrjagin on the ring of intersections in the space of planes with a common point. In fact, the number of closed geodesics on a manifold homeomorphic to an n-dimensional sphere is equal to $2n - 1$. First of all we define in the space P_n of rectifiable curves on the sphere S_n a certain $(n-1)$-dimensional ∇-cycle.

Suppose that S_n is the unit sphere in the space E_{n+1} of $(n+1)$ dimensions. Suppose that $q_1, q_2, \cdots, q_{n+1}$ are $(n+1)$ curves of P_n, where $\rho(q_i, q_j) < \epsilon$, $i, j = 1, 2, \cdots, n+1$. We stretch on the curves $q_1, q_2, \cdots, q_{n+1}$ on the sphere S_n an n-dimensional "sausage" $T_n(q_1, q_2, \cdots, q_{n+1})$ (material support of the $(n-1)$-dimensional simplex in P_n with vertices q_1, $q_2, \cdots, q_n$). To each curve q of P_n we assign a two-dimensional film in E_{n+1}, stretched on q, in fact by joining the center of gravity of q in E_{n+1} with all the points of q by rectilinear segments. The family of such films, stretched on all the curves of the simplex $T_n(q_1, q_2, \cdots, q_{n+1})$ forms an $(n+1)$-dimensional band. Define a function $f_n(q_1, q_2, \cdots, q_n)$ as the degree of covering by this band, modulo 2, of the point 0, the center of the sphere S_n in E_{n+1}. The sequence f_n defines a certain $(n-1)$-dimensional ∇-cycle, which we shall denote by Y. The support of the cycle Y is all the curves on which the films passing through 0 are stretched. This support on intersecting with the $3(n-1)$-dimensional Δ-cycle $A_{3(n+1)}$ yields a $2(n-1)$-dimensional cycle A_{2n-2}—the family of all great circles.

L. S. Pontrjagin proved that the length of the space A^{2n-2} for $n = 2^k$ is equal to $2n-2$, in fact that there exist $2n-2$ one-dimensional ∇-cycles, whose product is not equal to zero. But in A_{2n-2} there exist unique homology classes of Δ- and ∇-cycles, and these one-dimensional ∇-cycles are homologous to cycles induced in $A^{(n)}$ by the cycles Z_f^1 of the preceding section.

Thus,

$$A_{3n-3} \times (Y \times Z_1^{f_1} \times Z_2^{f_2} \times \cdots \times Z_1^{f_{2n-2}}) \neq 0.$$

The length of A_{3n-3} is equal to $2n-1$.

Repeating the considerations of the preceding section, we obtain $2n-1$ classes of Δ-cycles subordinated to one another, specifically, one class $[y_0]$ of $(n-1)$-dimensional Δ-cycles such that $y \times Y \neq 0$ (as an example of such a class one may choose the collection of all circumferences passing through two given points on S_n which are not diametrically opposed), and classes $[y_i]$, $i = 1, 2, \cdots, 2n-2$, all $(n+i-1)$-dimensional Δ-cycles such that $y_i \times [Y \times Z_{f_1}^1 \times Z_{f_2}^1 \times \cdots \times Z_{f_i}^1] \neq 0$. Evidently the class $[y_i]$ is subordinated to the class $[y_{i+k}]$ by means of a class of k-dimensional ∇-cycles obtained by multiplying the k-cycles Z_f^1.

Hence, if c_i, for $i = 0, 1, 2, \cdots, 2n-2$, are the minimum maxima of the function $y(q)$ on the cycles y_i of the class $[y_i]$, then, as in the preceding section, $0 < c_0 < c_1 < \cdots \leqq c_{2n-2}$. To each c_i there corresponds a closed geodesic of length c_i. If $c_i = c_{i+1} = c$, there exists a k-dimensional system of closed geodesics of length c.

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